On a class of commutative groupoids determined by their associativity triples

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Abstract. Let $G = G(\cdot)$ be a commutative groupoid such that $\{(a,b,c) \in G^3;\ a \cdot bc \neq ab \cdot c\} = \{(a,b,c) \in G^3;\ a = b \neq c \text{ or } a \neq b = c\}$. Then G is determined uniquely up to isomorphism and if it is finite, then $\mathrm{card}(G) = 2^i$ for an integer $i \geq 0$.

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For a groupoid $G = G(\cdot)$ denote by $\mathbf{Ns}(G)$ the set of its non-associative triples, i.e. $\mathbf{Ns}(G) = \{(a,b,c) \in G^3;\ a \cdot bc \neq ab \cdot c\}$. If $\mathcal{V}$ is a variety of groupoids and S a non-empty set, then it can be a non-trivial problem to determine all such $N \subseteq S^3$ that $N = \mathbf{Ns}(G)$ for a groupoid $G = S(\cdot) \in \mathcal{V}$. For example, it is known [1], [2] that $\mathbf{Ns}(G) \neq \{(a,a,a);\ a \in G\}$ for any non-empty groupoid G.

In the present short note we investigate the case when $\mathcal{V}$ is the variety of the commutative groupoids and $\mathbf{Ns}(G) = \{(a,b,c) \in G^3;\ a = b \neq c \text{ or } a \neq b = c\}$. We shall show that all such non-trivial groupoids can be obtained by a slight modification of a 2-elementary Abelian group and that these groupoids are determined up to isomorphism by $\mathrm{card}(G)$. Moreover, whenever G is finite and non-trivial, then $\mathrm{card}(G) = 2^i$ for an integer $i \geq 1$.

Note that $a \cdot ba = ab \cdot a$ for any $a,b \in G$ whenever G is a commutative groupoid. The set $\{(a,b,c) \in G^3;\ a = b \neq c \text{ or } a \neq b = c\}$ thus covers all $(a,b,c) \in G^3$ such that $\mathrm{card}\{a,b,c\} \leq 2$ and $a \cdot bc \neq ab \cdot c$ can occur.

Theorem 1. *For an Abelian group $G(+)$ and each $0 \neq e \in G$ define on the set G a commutative groupoid G_e by $0 \cdot 0 = e$, $a \cdot b = a + b$ and $a \cdot 0 = 0 \cdot a = 0$ for any $a,b \in G \backslash \{0\}$. If $G(+)$ is 2-elementary, then $\mathbf{Ns}(G_e) = \{(a,b,c) \in G^3;\ a = b \neq c$ or $a \neq b = c\}$. Conversely, if $G(\cdot)$ is a commutative groupoid where $a \cdot bc \neq ab \cdot c$ if and only if $a = b \neq c$ or $a \neq b = c$, and $\mathrm{card}(G) > 1$, then there exist a 2-elementary Abelian group $G(+)$ and an element $0 \neq e \in G$ such that $G(\cdot) = G_e$. Moreover, G_e is isomorphic to G_f for any choice of $e,f \in G$, $e \neq 0 \neq f$.*

Proof: Only the converse part of the theorem requires a proof. Let us hence assume that $G(\cdot)$ is a commutative groupoid, $\mathrm{card}(G) > 1$ and $\mathbf{Ns}(G(\cdot)) = \{(a,b,c) \in G^3;\ a = b \neq c \text{ or } a \neq b = c\}$. As G is commutative, we have

(1) $\quad a \cdot ba = ab \cdot a$ for any $a,b \in G$.

Let $a = bc$, where $a, b, c \in G$ are pair-wise distinct. If $c \neq ab$, then $aa \cdot b = (bc \cdot a) \cdot b = (b \cdot ca) \cdot b = b(ca \cdot b) = b(c \cdot ab) = bc \cdot ab = a \cdot ab$. Hence $c = ab$ and we have

(2) If $a = bc$, $b \neq a \neq c$ and $b \neq c$, then $b = ac$ and $c = ab$.

Further, we shall prove

(3) If $a = bc$, $b \neq a \neq c$ and $b \neq c$, then $a^2 = b^2 = c^2$ and $a^2 \notin \{a, b, c\}$.

To see this, observe that $c^2 = ab \cdot c = a \cdot bc = a^2$ by (2) and that $a^2 = a$ implies $a \cdot bb = a \cdot aa = a = cb = ab.b$.

If $a \in G$ is such that a, a^2, a^3 are pair-wise distinct, we obtain from (3) $a^2 \notin \{a^3, a^2, a\}$, a contradiction. Therefore it holds

(4) $a = a^2$ or $a^2 = a^3$ or $a = a^3$ for any $a \in G$.

Let $a, b, c \in G$ be again pair-wise distinct and $a = bc$. Then $a \neq a^2$ by (3), and $a^3 = a$ implies $a \cdot bb = a \cdot aa = a^3 = a = c \cdot b = ab \cdot b$. Hence we have

(5) If $a = bc$, $b \neq a \neq c$ and $b \neq c$, then $a \cdot a^2 = b \cdot a^2 = c \cdot a^2 = a^2 = b^2 = c^2$.

We shall now order the set G by $a < b$ iff $ab = b$ and $a \neq b$. From $a < b$ and $b < a$ it follows $b = ab = ba = a$ and from $a < b < c$ we obtain $ac = a \cdot bc = ab \cdot c = bc = c$. Therefore $<$ really is a (sharp) ordering of G.

Let again $a, b, c \in G$ be pair-wise distinct and with $a = bc$. If $e < a$, then $ec = e \cdot ab = ea \cdot b = ab = c$. Consequently, we have

(6) Let $a = bc$, $b \neq a \neq c$ and $b \neq c$. If $e < a$, then $e < b$ and $e < c$.

Conversely, suppose that $a < e$. Then $b \neq e \neq c$, $eb = ea \cdot b = e \cdot ab = ec$ and $eb \cdot c = e \cdot bc = ea = e$. From $eb = c$ it follows $ec = c$, $e < c$ and by (2) and (6) $e < a$. Therefore $eb \neq c$. If eb, c, e are pair-wise distinct, then $a < e$ implies by (6) that $a < c$, a contradiction. It follows $eb = e$ and we obtain

(7) Let $a = bc$, $b \neq a \neq c$ and $b \neq c$. If $a < e$, then $b < e$ and $c < e$.

For $a, b \in G$ put $(a, b) \in r$ iff $a \neq b$ and $a \neq ab \neq b$, and denote by $\sim$ the least equivalence containing the relation r. From (6) and (7) we get by induction immediately

(8) Let $a, b, e \in G$ and let $a \sim b$. Then $a < e$ iff $b < e$, and $e < a$ iff $e < b$.

Denote by $\mathcal{E}$ the set of equivalence classes of $\sim$. By the definitions of $\sim$ and $<$ we have either $a \sim b$, or $a < b$, or $b < a$ for any $a, b \in G$. Hence it follows from (8) that $<$ induces a linear ordering of $\mathcal{E}$. Suppose that $(\mathcal{E}, <)$ has no maximum element. Then for $a \in G$ we can choose $b \in G$ with $a < b$, $a^2 < b$. Then $b \cdot aa = b \cdot a^2 = b = ba = ba \cdot a$, a contradiction.

Let $U \in \mathcal{E}$ be the maximum element of $(\mathcal{E}, <)$ and suppose that $a, b \in U$, $a \neq b$. Then $a \neq ab \neq b$, and we obtain $a^2 > a$ by (3) and (5). This is a contradiction, and hence U contains exactly one element, say u.

For $u \neq a \in G$ we have $ua = u$, and thus $u = ua \cdot a \neq u \cdot a^2$ provides $a^2 = u$. Therefore $a < b < u$ would imply $a \cdot bb = au = u = bb = ab \cdot b$, which contradicts our hypothesis. It follows that the equivalence $\sim$ has exactly two classes and by (7) we have $ab \notin \{a, b, u\}$ for any $a, b \in G$, $a \neq b$, $a \neq u \neq b$. Moreover, $u^2 = aa \cdot u \neq a \cdot au = u$.

Put now $u = 0$ and define $G(+)$ by $a+0 = 0+a = a$ for any $a \in G$ and $a+b = ab$ for $a, b \in G$, $a \neq 0 \neq b$. Clearly, $a + (b + c) = (a + b) + c$ whenever $0 \in \{a, b, c\}$, and by (2) also when $a = b$ or $b = c$. Similarly, $a+(b+ab) = a+a = ab+ab = ab+(a+b)$ for $a, b \in G$, $a \neq b$, $a \neq 0 \neq b$. Finally, $a+(b+c) = a+bc = a \cdot bc = ab \cdot c = ab+c = (a + b) + c$ when $a, b, c \in G$ are pair-wise distinct and $c \neq ab$. It follows that $G(+)$ is a 2-elementary Abelian group and we see that $G(\cdot) = G_{u^2}$. $\square$

References

[1] Drápal A., Kepka T., *Sets of associative triples*, Europ. J. Combinatorics **6** (1985), 227–231.

[2] Drápal A., *Groupoids with non-associative triples on the diagonal*, Czech. Math. Journal **35** (1985), 555–564.

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