The existence of local homeomorphisms of degree $n > 1$ on local dendrites

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Abstract. In this paper we characterize local dendrites which are the images of themselves under local homeomorphisms of degree n for each positive integer n.

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It is shown by Maćkowiak [1] that each local homeomorphism of a continuum onto a tree-like continuum is a homeomorphism. In [3], we have shown that each nonunicoherent continuum is the image of some continuum under local homeomorphisms of degree n for each positive integer n. However, we do not know if there is a local homeomorphism of degree $n > 1$ from a continuum onto a unicoherent continuum.

A related problem is the following: Which continua are the images of themselves under local homeomorphisms of degree n for each (or for some) positive integer $n > 1$? In this paper, as a partial answer to the question, we characterize local dendrites which are the images of themselves under local homeomorphisms of degree n for each positive integer n.

A continuous surjection f between spaces X and Y is said to be:

(1) a local homeomorphism if for each point x in X there is an open neighborhood U of x such that f is a homeomorphism on U and $f(U)$ is open;
(2) of degree n if $f^{-1}(y)$ has exactly n points for each $y \in Y$.

A local dendrite is any Peano continuum which contains at most a finite number of simple closed curves. A local dendrite is called a dendrite if it contains no simple closed curve. A local dendrite is called a graph if it contains at most a finite number of end points.

Theorem. *Let X be a local dendrite and let $n > 1$ be an integer. Then, X is the image of itself under a local homeomorphism of degree n if and only if X consists of dendrites $D_1, D_2, \dots, D_k$, where $k \geq 3$, such that for each i and j less than $k+1$*

(i) *D_i intersects D_j whenever either $|i-j| \leq 1$ or $|i-j| = k-1$, and*
(ii) *there are homeomorphisms $h_{i,j} : D_i \to D_j \cup D_{j+1}$, where $D_{k+1} = D_1$, such that if S is a simple closed curve in X, then $h_{i,j}(S \cap D_i) = S \cap (D_j \cup D_{j+1})$.*

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Proof: We first assume that X consists of dendrites $D_1, \dots D_k$, $k \geq 3$, satisfying (i) and (ii), and we will show that there is a map $h : X \to X$ which is a local homeomorphism of degree n. (The construction runs similarly to the construction of the map f in the proof of the theorem in [3].) Put $A = D_1$, $B = D_2 \cup \dots \cup D_k$, $P = D_1 \cap D_k$ and $Q = D_1 \cap D_2$. For each $i = 1, 2, \dots, n$, let A_i and B_i be distinct copies of A and B, respectively; in each A_i (B_i, respectively) we distinguish copies P_{A_i} and Q_{A_i} (P_{B_i} and Q_{B_i}, respectively) of P and Q. Consider the free union

$$U = \oplus\{A_i \oplus B_i : \; i = 1, 2, \dots, n\}$$

and let ϱ be an equivalence relation on U which identifies only the following sets: P_{A_i} with P_{B_i} for each $i = 1, 2, \dots, n$; $Q_{A_{i+1}}$ with Q_{B_i} for each $i = 1, 2, \dots, n-1$; and Q_{B_n} with Q_{A_1}.

Note that the quotient space $Y = U/\varrho$ is a continuum in the adjunction topology which consists of copies A_i^Y and B_i^Y of the sets A_i and B_i, respectively.

Let $h_1 : Y \to X$ be the map which identifies each A_i^Y with A and each B_i^Y with B. It is not difficult to check that h_1 is a local homeomorphism of degree n. Now we show that there is a homeomorphism h_2 from X onto Y.

Indeed, by (i), X contains a simple closed curve S. Put $D'_i = D_i \setminus (D_{i-1} \cup D_{i+1})$. By (ii), since the homeomorphic image of a dendrite is a dendrite, $D_j \cup D_{j+1}$ is a dendrite, and consequently $X \setminus D'_i$ is also a dendrite. Therefore, S is the only simple closed curve in X. Consider the arcs $a_i b_i = D_i \cap S$ and $a_{i+1} b_{i+1} = D_{i+1} \cap S$, where $b_i \in D_{i+1}$ and $b_{i+1} \in D_{i+2}$. Note that there are homeomorphisms $H_i : D_i \to D_{i+1}$ such that $H_i(a_i) = a_{i+1}$ and $H_i(b_i) = b_{i+1}$. Hence there are homeomorphisms from D_i onto $X \setminus D'_i$ which identify the arcs $S \cap D_i$ and $S \cap (X \setminus D'_i)$. Further, by the construction, $A_1^Y, \dots, B_n^Y$ also satisfy (i) and (ii). Hence Y contains only one simple closed curve C and there are homeomorphisms from A_i^Y onto $Y \setminus A_i'^Y$, where $A_i'^Y = A_i^Y \setminus (B_{i-1}^Y \cup B_i^Y)$ which identify the arcs $C \cap A_i^Y$ and $C \cap (Y \setminus A_i'^Y)$. Thus since A_i^Y and B_i^Y are copies of A and B, respectively, X is homeomorphic to Y, and the existence of h_2 is proved. Finally, we are putting $h = h_1 h_2$. This concludes the first part of the theorem.

We now assume that f is a local homeomorphism of degree $n > 1$ from X onto itself. Since each local homeomorphism of a dendrite is a homeomorphism, X contains a simple closed curve S. Let G be a maximal (with respect to inclusion) subcontinuum of X which contains no end points of itself. We see that G is a graph which contains S.

We show that $f(G) = G$. For a start, suppose that $f(G)$ is not in G. Then there exists a component C of $X \setminus G$ which intersects $f(G)$. Thus, since C contains no simple closed curves, $f(G)$ contains an end point e which is in C. Let $p \in G$ be a point such that $e = f(p)$. Take an open neighborhood U of p that exists by the definition of the local homeomorphism f and choose two points p_1 and p_2 in $G \cap U$ both different from p and such that p lies in the arc $p_1 p_2 \subset G \cap U$. Then the arc $p_1 p_2$ is mapped homeomorphically under f onto the arc $f(p_1) f(p_2)$ which contains the point e with $f(p_1) \neq e \neq f(p_2)$, whence e cannot be an end point of $f(G)$, a contradiction. Thus $f(G) \subset G$. Next observe that the image of every component of $X \setminus G$ under f is again a component of $X \setminus G$. To prove this it is enough to show that for each component C of $X \setminus G$ we have $f(G) \cap G = \emptyset$. Suppose that $f(C) \cap G \neq \emptyset$ for some C, and let K be a component of $f^{-1}(G) \cap \overline{C}$ with $K \cap C \neq \emptyset$. Then K is a dendrite. Take an end point e of K which is in C. It is not difficult to find an arbitrarily small open set V of e in X such that $f(V)$ is not open in X. Therefore, $f(X \setminus G) \subset X \setminus G$, and consequently $f(G) = G$.

We also show that $G = S$. In fact, since f is a local homeomorphism, $\operatorname{ord}_x X = \operatorname{ord}_{f(x)} X$ for each point $x \in X$ (where $\operatorname{ord}_z Z$ denotes the Menger-Urysohn order of a point z in a space Z). In particular, since $f(G) = G$, we have $\operatorname{ord}_r G = \operatorname{ord}_{f(r)} G$ for each $r \in R(G)$, where $R(G)$ denotes the set of all ramification points of G. $R(G)$ is at most finite, because G is a graph. Hence, since f is of degree n, we have $n \cdot \operatorname{card} R(f(G)) = \operatorname{card} R(G)$. On the other hand, since $f(G) = G$, we have $\operatorname{card} R(f(G)) = \operatorname{card} R(G)$. Whence $\operatorname{card} R(G) = 0$. Therefore, G is composed of points of order 2 exclusively. Whence G is just the simple closed curve S.

Further, the map f being a local homeomorphism, it is a covering projection. Hence, since $f(S) = S$ (because $G = S$), the map $f \mid S : S \to S$ is topologically equivalent to the map $z \mapsto z^n$ on the unit circle $S^1 = \{z \in R^2 : |z| = 1\}$. It is known ([4, Remark, p. 2]) that $f \mid S$ has the unique fixed point q_1.

Now, we construct the dendrites $D_1, D_2, \dots, D_k$ as follows. Consider the mapping $f^2 : X \to X$. Then $(f^2)^{-1}(q_1) = \{q_1, q_2, \dots, q_{n^2}\}$, where $q_1, q_2, \dots, q_{n^2}$ are cyclically ordered as indicated. Note that $f^{-1}(q_1)$ consists of points q_{sn-n+1} for $s = 1, \dots, n$.

Further, let A_i be an arc $q_i q_{i+1}$ in S, where $q_{n^2+1} = q_1$ such that $A_i \cap (f^2)^{-1}(q_1) = \{q_i, q_{i+1}\}$. Put $B_i = S \setminus A_i$. Define D_i to be the component of $X \setminus B_i$ which contains A_i. Clearly, $X = D_1 \cup D_2 \cup \dots \cup D_{n^2}$ and $D_1, D_2, \dots, D_{n^2}$ satisfy (i). We show that they also satisfy (ii). In fact, it follows from the construction that for every j between 1 and n

$$f^2(D_i) = X = f(D_j \cup \dots \cup D_{j+n-1}), \text{ where } D_m = D_{(m \mod n^2)} \text{ for } m > n^2,$$
$$f^2(S \cap D_i) = S = f(S \cap (D_j \cup \dots \cup D_{j+n-1})).$$

This implies that there are homeomorphisms from D_i onto $D_j \cup D_{j+1}$ which identify the arcs $S \cap D_i$ and $S \cap (D_j \cup D_{j+1})$. This completes the proof of the theorem. □

Corollary 1. *Let X be a local dendrite and suppose that there is a local homeomorphism of degree $n > 1$ from X onto itself. Then there are local homeomorphisms of degree n from X onto itself for each positive integer n.*

Corollary 2. *Let X be a graph and let $2 \leq n < \infty$. Then X is the image of itself under a local homeomorphism of degree n if and only if X is a simple closed curve.*

Proof: Since the set of all ramification points of a graph is at most finite, we conclude that the only graph satisfying (i) and (ii) of the theorem is the simple closed curve. Therefore, the theorem implies the corollary, because every graph is a local dendrite. □

Remark 1. The local dendrites which are the images of themselves under local homeomorphisms of degree n for each positive integer n have "nice" decompositions. Therefore, we can see their geometrical structure. Thus, every one of them can be constructed as the local dendrite X in the example in [2].

Remark 2. We see that the Sierpiński curve, the Menger curve, a torus, an annulus satisfy (i) and (ii) of the theorem. We see also that they are the images of themselves under local homeomorphisms of degree n for each positive integer n. However, we do not know for what wider classes of continua the theorem is true.

References

[1] Maćkowiak T., *Local homeomorphisms onto tree-like continua*, Colloq. Math. **38** (1977), 63–68.

[2] Miklos S., *Exactly* $(n, 1)$ *mappings onto generalized local dendrites*, Topology Appl. **31** (1989), 47–53.

[3] ———, *Local homeomorphisms onto nonunicoherent continua*, Period. Math. Hungar. **20** (1989), 305–306.

[4] Rosenholtz I., *Local expansions, derivatives, and fixed points*, Fund. Math. **91** (1976), 1–4.

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