Contact manifolds, harmonic curvature tensor and (k, μ)-nullity distribution

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Abstract. In this paper we give first a classification of contact Riemannian manifolds with harmonic curvature tensor under the condition that the characteristic vector field ξ belongs to the (k, μ)-nullity distribution. Next it is shown that the dimension of the (k, μ)-nullity distribution is equal to one and therefore is spanned by the characteristic vector field ξ.

Keywords: contact Riemannian manifold, harmonic curvature, D-homothetic deformation

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It is well known that there exist contact Riemannian manifolds $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ for which the curvature tensor R in the direction of the characteristic vector field ξ satisfies $R_{XY}\xi = 0$, for any tangent vector fields X, Y of M^{2n+1}. The tangent sphere bundle of a flat Riemannian manifold, for example, admits such a structure [2]. Applying a D-homothetic deformation [7] on M^{2n+1} with $R_{XY}\xi = 0$, we find a new class of contact metric manifolds satisfying the relation

$$(1.1) \qquad R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad (k, \mu) \in \mathbb{R}^2$$

where $2h$ is the Lie derivative of φ with respect to ξ. An interesting property of this class is that the form of (1.1) is invariant under a D-homothetic deformation.

The purpose of this paper is, on the one hand, the classification of the contact Riemannian manifolds having a harmonic curvature tensor under the condition that the characteristic vector field ξ belongs to the (k, μ)-nullity distribution, i.e. satisfies the condition (1.1), and on the other hand, to prove that the (k, μ)-nullity distribution, which we will denote by $N(k, \mu)$ for $k < 1$, $k \neq 0$, is a 1-dimensional subspace of T_pM for every $p \in M$ and is spanned by the characteristic vector field ξ.

2. Preliminaries and known results.

Manifolds and tensor fields are supposed to be of the class C^∞.

Let $M = M^{2n+1}$ be a connected differentiable manifold with contact form η, i.e. a tensor field of type $(0, 1)$ satisfying $\eta \wedge (d\eta)^n \neq 0$. It is well known that such a manifold admits a vector field ξ, called the *characteristic vector field* such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$, for every $X \in \chi(M)$ ($\chi(M)$ being the Lie algebra of the

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vector fields of M). Moreover, M admits a Riemannian metric g and a tensor field φ of type (1.1) such that

$$\text{(2.1)} \qquad \text{(i) } \varphi^2 = -I + \eta \otimes \xi, \ \text{(ii) } g(X, \xi) = \eta(X), \ \text{(iii)} g(X, \varphi Y) = d\eta(X, Y).$$

We then say that (φ, ξ, η, g) is a *contact metric structure*. As a consequence of these relations, one has

$$\text{(2.2)} \qquad \text{(i) } g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ \text{(ii) } \varphi\xi = 0, \ \text{(iii) } \eta\varphi = 0.$$

Denoting by $\mathcal{L}$ and R the Lie differentiation and the curvature tensor respectively, we define the operators ℓ and h by

$$\text{(2.3)} \qquad \text{(i) } \ell X = R(X, \xi)\xi, \ \text{(ii) } hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)X.$$

The $(1, 1)$ tensors ℓ and h are self-adjoint and satisfy

$$\text{(2.4)} \qquad \text{(i) } h\xi = 0, \ \text{(ii) } \ell\xi = 0, \ \text{(iii) } tr\ h = tr\ h\varphi = 0, \ \text{(iv) } h\varphi = -\varphi h.$$

Since h anticommutes with φ, if X is an eigenvector of h corresponding to the eigenvalue λ, then φX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. If $\triangledown$ is the Riemannian connection of g, then

$$\text{(2.5)} \qquad \text{(i) } \triangledown_X \xi = -\varphi X - \varphi h X, \ \text{(ii)} \triangledown_X \varphi = 0, \ \text{(iii) } \varphi \ell \varphi - \ell = 2(h^2 + \varphi^2).$$

A contact metric manifold for which ξ is a Killing vector field is called a K*-contact* manifold. It is well known that a contact manifold is K-contact if and only if $h = 0$. Moreover, on a K-contact manifold it is valid $R(X, \xi)\xi = X - \eta(X)\xi$. A contact metric manifold is said to be a *Sasakian* manifold if

$$\text{(2.6)} \qquad (\triangledown_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

in which case

$$\text{(2.7)} \qquad \text{(i) } \triangledown_X \xi = -\varphi X, \ \text{(i) } R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Note that a Sasakian manifold is K-contact, but the converse holds if and only if $\dim M = 3$.

A contact manifold is said to be η*-Einstein* if

$$\text{(2.8)} \qquad Q = a\, I\, d + b\eta \otimes \xi,$$

where Q is the Ricci operator and a, b are smooth functions on M. The sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a ξ*-sectional curvature*, while the sectional curvature $K(X, \varphi X)$ is called a φ*-sectional curvature*. The (k, μ)*-nullity* distribution of a contact metric manifold for the pair $(k, \mu) \in \mathbb{R}^2$, is a distribution

$$\begin{aligned} N(k, \mu) : p \to N_p(k, \mu) = \{Z \in T_pM \mid R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &+ \mu[g(Y, Z)hX - g(X, Z)hY]\}. \end{aligned}$$

So, if the characteristic vector field ξ belongs to the (k, μ)-nullity distribution we have

$$\text{(2.9)} \qquad R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

Now the following lemma is well known [4], but for completness, we also give the proof.

Lemma 2.1. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then*
(2.10)
1. $\ell X = k(X-\eta(X)\xi)+\mu hX,\ \forall\, X\in\chi(M)$
2. $R(\xi,X)Y = k(g(X,Y)\xi-\eta(Y)X)+\mu(g(hX,Y)\xi-\eta(Y)hX)$
3. $h^2=(k-1)\varphi^2,\ k\le 1$
4. $QX=[2(n-1)-n\mu]X+[2(n-1)+\mu]hX+[2(1-n)+n(2k+\mu)]\eta(X)\xi,\quad n\ge 1$
5. $\varphi Q = Q\varphi - 2[2(n-1)+\mu]h\varphi.$

Proof: 1. Using the relations (2.3 (i)) and (2.9) we have

$$\begin{aligned}\ell X &= R(X,\xi)\xi = k(\eta(\xi)X-\eta(X)\xi)+\mu(\eta(\xi)hX-\eta(X)h\xi)\\ &= k(X-\eta(X)\xi)+\mu hX.\end{aligned}\tag{2.11}$$

2. Using the relation (2.9) and $g(hX,Y)=g(X,hY)$ we have

$$\begin{aligned}g(R(\xi,X)Y,Z) &= g(R(Y,Z)\xi,X) = g(k(\eta(Z)Y-\eta(Y)Z),X)+g(\mu(\eta(Z)hY\\ &\quad -\eta(Y)hZ),X) = k[g(X,Y)\eta(Z)-g(X,Z)\eta(Y)]+\mu[g(X,hY)\eta(Z)\\ &\quad -g(X,hZ)\eta(Y)] = k[g(X,Y)g(\xi,Z)-\eta(Y)g(X,Z)]\\ &\quad +\mu[g(hX,Y)g(\xi,Z)-\eta(Y)g(hX,Z)]\end{aligned}$$

and since this equation is valid for any $Z\in\chi(M)$, we get the required result.

3. Using (2.5 (iii)), (2.10 (i)), and (2.4 (iv)) we have

$$\begin{aligned}(-\ell+\varphi\ell\varphi)X &= -\ell X+\varphi\ell\varphi X\\ &= -k(X-\eta(X)\xi)-\mu hX+\varphi(k\varphi X+\mu h\varphi X)\\ &= 2k\varphi^2X-\mu h(X+\varphi^2X)=2k\varphi^2X\end{aligned}$$

but on the other hand, $-\ell+\varphi\ell\varphi = 2(h^2+\varphi^2)$, so we easily get the result. Now using the definition of the Ricci operator Q and the orthonormal basis $\{e_i\}$ one easily computes that

$$Q\xi=\sum_{i=1}^{2n+1}R(\xi,e_i)e_i=(2n+1)k\xi-k\xi+\mu(tr\ h)\xi=2nk\xi.$$

But on any contact manifold $Q(\xi,\xi)=2n-\|h\|^2$, hence we have $\|h\|^2=2n(1-k)\ge 0$, from which $k\le 1$.

4.–5. Similarly, one can easily prove these cases as well. $\square$

For more details concerning contact metric manifolds we refer the reader to [2].

We close this section with a brief discussion of the harmonicity of the curvature tensor of a Riemannian manifold. It is well known that, if the divergence of the curvature tensor of a Riemannian manifold is equal to zero, then this curvature tensor is called harmonic. So, a Riemannian manifold has harmonic curvature tensor if and only if the Ricci operator Q, which is given by $g(QX,Y) = S(X,Y)$ where S is the Ricci tensor, satisfies the following relation:

$$(\nabla_X Q)Y - (\nabla_Y Q)X = 0. \tag{2.12}$$

3. Contact manifolds with harmonic curvature tensor and ξ belonging to the (k,μ)-nullity distribution.

Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution, i.e.

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad (k,\mu) \in \mathbb{R}^2. \tag{3.1}$$

Let Q be the Ricci operator of M, then the manifold has the harmonic curvature tensor if, as mentioned above,

$$(\nabla_X Q)Y - (\nabla_Y Q)X = 0 \tag{3.2}$$

for any vector fields X,Y of M.

We first prove the following lemma.

Lemma 3.1. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution. Then*

$$\begin{aligned} g((\nabla_X Q)Y - (\nabla_Y Q)X,\xi) = {} & 2[2(n+k-1) - \mu(k-1)]g(X,\varphi Y) \\ & + 2g(Y,Q\varphi X) - 2[2(n-1)+\mu]g(Y,h\varphi X) \\ & + g(Y,(Q\varphi h + hQ\varphi)X) \end{aligned} \tag{3.3}$$

for any $X,Y \in \chi(M)$.

Proof: Using the symmetry of the operator $\nabla_X Q$ and $(2.10,4)$ we have

$$g((\nabla_X Q)Y,\xi) = g(Y,(\nabla_X Q)\xi) = -2nkg(Y,\varphi X + \varphi hX) + g(Y,Q(\varphi X + \varphi hX)).$$

Similarly,

$$g((\nabla_Y Q)X,\xi) = -2nkg(X,\varphi Y + \varphi hY) + g(X,Q(\varphi Y + \varphi hY)).$$

Hence

$$\begin{aligned} g((\nabla_X Q)Y - (\nabla_Y Q)X,\xi) = {} & 4nkg(X,\varphi Y) \\ & + g(Y,Q\varphi X) + g(Y,Q\varphi hX) \\ & + g(Y,\varphi QX) + g(Y,h\varphi QX). \end{aligned} \tag{3.4}$$

Now using $(2.10,5)$ and $(2.10,3)$ we have

$$\begin{aligned} g((\nabla_X Q)Y-(\nabla_Y Q)X,\xi) &= 4nkg(X,\varphi Y)+g(Y,Q\varphi X)+g(Y,Q\varphi hX)\\ &\quad + g(Y,Q\varphi X-2[2(n-1)+\mu]h\varphi X)\\ &\quad + g(Y,hQ\varphi X-2[2(n-1)+\mu](k-1)\varphi^3 X)\\ &= 2[2(k+n-1)-\mu(k-1)]g(X,\varphi Y)+2g(Y,Q\varphi X)\\ &\quad - 2[2(n-1)+\mu]g(Y,h\varphi X)+g(Y,(Q\varphi h+hQ\varphi)X) \end{aligned}$$

and the proof is complete. □

We now state the main result.

Theorem 3.1. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with harmonic curvature tensor and ξ belonging to the (k,μ)-nullity distribution. Then M is either*

(i) *an Einstein Sasakian manifold, or*
(ii) *an η-Einstein manifold, or*
(iii) *locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature equal to 4, including a flat contact metric structure for $n=1$.*

The proof of this theorem depends largely on the following results.

Lemma 3.2 [4]**.** *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then $k\le 1$. If $k<1$, then M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$, $D(-\lambda)$ defined by the eigenspaces of h, where $\lambda=\sqrt{1-k}>0$.*

Theorem 3.2 [2]**.** *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with $R_{XY}\xi=0$ for all vector fields X,Y of M. Then M is locally the product of a flat $(n+1)$-dimensional manifold of positive constant curvature equal to 4, including a flat contact metric structure for $n=1$.*

Theorem 3.3 [4]**.** *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. If $k<1$ then for any X orthogonal to ξ*

(1) *The ξ-sectional curvature $K(X,\xi)$ is given by*

$$K(X,\xi)=\begin{cases} k+\lambda\mu, & \text{if } X\in D(\lambda)\\ k-\lambda\mu, & \text{if } X\in D(-\lambda),\end{cases}$$

(2) *the sectional curvature of a plane section $\{X,Y\}$ normal to ξ is given by*

$$K(X,Y)=\begin{cases} 2(1+\lambda)-\mu, \text{ if } X,Y\in D(\lambda),\\ -(k+\mu)(g(X,\varphi Y))^2, \text{ for any unit vectors } X\in D(\lambda),\ Y\in D(-\lambda)\\ 2(1-\lambda)-\mu, \text{ if } X,Y\in D(-\lambda),\ n>1.\end{cases}$$

Next we prove the following lemma.

Lemma 3.3. *Let* $[M^{2n+1},(\varphi,\xi,\eta,g)]$ *be a contact metric manifold with* ξ *belonging to the* (k,μ)*-nullity distribution. Then*

$$\text{(i) If } X\in D(\lambda),\ h(\nabla_\xi X)=\lambda(\nabla_\xi X+\mu\varphi X) \tag{3.5}$$

$$\text{(ii) If } X\in D(-\lambda),\ h(\nabla_\xi X)=-\lambda(\nabla_\xi X+\mu\varphi X). \tag{3.6}$$

Proof: (i) Since $X\in D(\lambda)$, applying (3.1) we easily get

$$R(\xi,X)\xi=-(k+\lambda\mu)X. \tag{1}$$

On the other hand, using the definition of the curvature tensor we have

$$\begin{aligned}R(\xi,X)\xi&=\nabla_\xi\nabla_X\xi-\nabla_{[\xi,X]}\xi=-\nabla_\xi(\varphi X+\varphi hX)\\&\quad+\varphi[\xi,X]+\varphi h[\xi,X]=-\lambda\varphi\nabla_\xi X+\varphi h\nabla_\xi X+\varphi(\varphi X+\varphi hX)\\&\quad+\varphi h(\varphi X+\varphi hX)=-\lambda\varphi\nabla_\xi X+\varphi h\nabla_\xi X-(1-\lambda^2)X\end{aligned}$$

and since $k=1-\lambda^2$, we have

$$R(\xi,X)\xi=-\lambda\varphi\nabla_\xi X+\varphi h\nabla_\xi X-kX. \tag{2}$$

Now comparing (1) with (2) we get

$$-\lambda\varphi\nabla_\xi X+\varphi h\nabla_\xi X=-\lambda\mu X, \tag{3.7}$$

or applying with φ and using $h\xi=0$ and $g(\nabla_\xi X,\xi)=0$ we get the required result (3.5).

(ii) For $X\in D(-\lambda)$, again applying (3.1) we have

$$R(\xi,X)\xi=-(k-\lambda\mu)X. \tag{3}$$

On the other hand, using the definition of the curvature tensor we easily have

$$R(\xi,X)\xi=\lambda\varphi\nabla_\xi X+\varphi h\nabla_\xi X-kX. \tag{4}$$

So, comparing (3) and (4) we have

$$\varphi h\nabla_\xi X=\lambda(-\varphi\nabla_\xi X+\mu X)$$

and acting with φ we get

$$h(\nabla_\xi X)=-\lambda(\nabla_\xi X+\mu\varphi X)$$

and the proof is complete. □

We are now going to give the proof of the main Theorem 3.1.

Proof of Theorem 3.1: The case of $k = 1$, $\mu \in \mathbb{R}$ gives $\lambda = \sqrt{1-k} = 0$, or equivalently $h = 0$. So, $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$ and the manifold is a Sasakian. Now using Lemma 3.1 we easily get that this manifold with harmonic curvature tensor is an Einstein manifold. Let $k < 1$ and $\mu \in \mathbb{R}$, and suppose $X \in D(\lambda)$, $Y \in D(-\lambda)$. Then one easily proves that $g(Y, Q\varphi hX + hQ\varphi X) = 0$ and using the harmonicity of the curvature tensor, applying Lemma 3.1, we get

$$g(Q\varphi X, Y) = \{\lambda[2(n-1)+\mu] - \lambda^2\mu - 2(n-\lambda^2)\}g(X, \varphi Y). \tag{1}$$

Replacing Y by φZ $(Z \in D(\lambda))$ and using (2.2 (i)) and (2.10, 5) we deduce

$$g(QX, Z) = c_1 g(X, Z), \quad \forall\, X, Z \in D(\lambda), \tag{3.8}$$

where

$$c_1 = \lambda[2(n-1)+\mu] + \lambda^2\mu + 2(n-\lambda^2) = \text{ const.} \tag{3.9}$$

Next, replacing X by φW $(W \in D(-\lambda))$ in (1) and using (2.2 (i)) we get

$$g(QW, Y) = c_2 g(W, Y), \quad \forall\, Y, W \in D(-\lambda), \tag{3.10}$$

where

$$c_2 = -\lambda[2(n-1)+\mu] + \lambda^2\mu + 2(n-\lambda^2). \tag{3.11}$$

Now differentiating (2.10, 4) with respect to ξ and again using (3.8) we get

$$\begin{aligned} g((\nabla_\xi Q)X + Q(-\varphi X - \varphi hX), Z) + g(QX, -\varphi Z - \varphi hZ)\\ = c_1[-g(\varphi X + \varphi hX, Z) - g(X, \varphi Z + \varphi hZ)] \end{aligned}$$

or

$$\begin{aligned} g((\nabla_\xi Q)X, Z) - g(Q(\varphi X + \varphi hX), Z) - g(QX, \varphi Z + \varphi hZ)\\ = c_1[g(\varphi X + \varphi hX, Z) + g(X, \varphi Z + \varphi hZ)]. \end{aligned} \tag{3}$$

But one easily can prove that

$$g(\varphi X + \varphi hX, Z) = (1+\lambda)g(\varphi X, Z),\ g(X, \varphi Z + \varphi hZ) = -(1+\lambda)g(Z, \varphi X) \tag{4}$$

and

$$\begin{aligned} g(Q\varphi X + Q\varphi hX, Z) = (1+\lambda)g(Q\varphi X, Z),\\ g(QX, \varphi Z + \varphi hZ) = -(1+\lambda)g(\varphi QX, Z). \end{aligned} \tag{5}$$

So, the equation (3) is reduced to

$$g((\nabla_\xi Q)X, Z) = 0, \quad \forall\, X, Z \in D(\lambda). \tag{3.12}$$

Now, since the curvature tensor is harmonic, using (4) and (5) and $g(\varphi X, Z) = 0$, we have

$$0 = g((\nabla_\xi Q)X, Z) = g((\nabla_X Q)\xi, Z) = -2nkg(\varphi X + \varphi hX, Z) \\ + g[Q(\varphi X + \varphi hX), Z] = (1+\lambda)g(Q\varphi X, Z).$$

Hence, $g(\varphi X, QZ) = 0$ and also since $g(QZ, \xi) = 0$, we conclude from (3.8) and Lemma 3.2 that

$$QX = c_1 X, \quad \forall X \in D(\lambda). \tag{3.13}$$

Similarly, one can obtain

$$QX = c_2 X, \quad \forall X \in D(-\lambda). \tag{3.14}$$

Now differentiating (3.13) with respect to ξ we have

$$(\nabla_\xi Q)X + Q\nabla_\xi X = c_1 \nabla_\xi X, \quad \forall X \in D(\lambda). \tag{3.15}$$

Now suppose that

$$\nabla_\xi X = (\nabla_\xi X)_\lambda + (\nabla_\xi X)_{-\lambda}\,. \tag{6}$$

Using (3.15) and this equation, we have

$$(\nabla_X Q)\xi = (\nabla_\xi Q)X = -Q\nabla_\xi X + c_1 \nabla_\xi X \\ = -Q[(\nabla_\xi X)_\lambda + (\nabla_\xi X)_{-\lambda}] + c_1(\nabla_\xi X)_\lambda + c_1(\nabla_\xi X)_{-\lambda}\,.$$

But from (3.13) and (3.14) we have

$$Q(\nabla_\xi X)_\lambda = c_1(\nabla_\xi X)_\lambda,\ Q(\nabla_\xi X)_{-\lambda} = c_2(\nabla_\xi X)_{-\lambda}\,.$$

So,

$$(\nabla_X Q)\xi = (c_1 - c_2)(\nabla_\xi X)_{-\lambda}\,, \tag{3.16}$$

where

$$c_1 - c_2 = 2\lambda[2(n-1) + \mu]. \tag{3.17}$$

On the other hand,

$$(\nabla_X Q)\xi = 2nk\nabla_X \xi + Q(\varphi X + \varphi hX) = -2nk(\varphi X + \varphi hX) + (1+\lambda)Q\varphi X$$

and using (3.14), we have

$$(\nabla_X Q)\xi = (1+\lambda)(c_2 - 2nk)\varphi X. \tag{3.18}$$

Comparing (3.16), (3.17) and (3.18) we get

$$2\lambda[2(n-1)+\mu](\nabla_\xi X)_{-\lambda} = (1+\lambda)(c_2 - 2nk)\varphi X. \tag{3.19}$$

Now, if we substitute the equation (6) into equation (3.5) of Lemma 3.3, we easily deduce that

$$(\nabla_\xi X)_{-\lambda} = -\frac{\mu}{2}\varphi X.$$

Substituting this equation into equation (3.19) and using (3.11) we conclude either

$$\text{(i) } \mu + 2(n-1) = 0, \text{ or } \text{ (ii) } k = \mu. \tag{3.20}$$

If the first (i) equality holds, then applying Lemma 2.1, we conclude that the Ricci operator Q is given by

$$QX = 2(n^2-1)X + 2(1+nk-n^2)\eta(X)\xi \tag{3.21}$$

which is of the form (2.8) and therefore, the manifold M^{2n+1} is η-Einstein.

If the second (ii) equality holds, then from Theorem 3.3 we get for the ξ-sectional curvatures

$$K(X,\xi) = (1+\lambda)k, \ \forall X \in D(\lambda), \quad K(X,\xi) = (1-\lambda)k, \ \forall X \in D(-\lambda) \tag{3.22}$$

and for the sectional curvatures

$$\begin{aligned} &\text{(i) } K(X,Y) = 2(1+\lambda) - k = (1+\lambda)^2, \quad \forall X,Y \in D(\lambda), \\ &\text{(ii) } K(X,Y) = 2(1-\lambda) - k = (1-\lambda)^2, \quad \forall X,Y \in D(-\lambda), \\ &\text{(iii) } K(X,Y) = 2(\lambda^2-1)(g(X,\varphi Y))^2, \quad \forall X \in D(\lambda), \quad \forall Y \in D(-\lambda). \end{aligned} \tag{3.23}$$

On the other hand, another implication of $k = \mu$ may be taken from Lemma 2.1, and therefore, we get

$$QX = [2(n-1) - nk]X + \lambda[2(n-1)+k]X, \quad \forall X \in D(\lambda). \tag{3.24}$$

But, as we proved $QX = c_1 X$ for every X, so we will have

$$2n-2-nk+2(n-1)\lambda+\lambda(1-\lambda^2) = 2(n-1)\lambda+\lambda(1-\lambda^2)+\lambda^2(1-\lambda^2)+2n-2\lambda^2,$$

from which we get

$$\lambda^4 + (1+n)\lambda^2 - (2+n) = 0. \tag{3.25}$$

The only positive root of this equation is $\lambda = 1$ and since $k = 1 - \lambda^2$ (Lemma 3.2), we conclude that $k = \mu = 0$. Hence $R_{XY}\xi = 0$ for all vector fields X, Y. Now, the equation (3.23) gives (i) K(X,Y)=4, $\forall X, Y \in D(\lambda)$, or (ii) K(X,Y)=0, either $X, Y \in D(-\lambda)$ or $X \in D(\lambda)$, $Y \in D(-\lambda)$. Therefore, we conclude that the manifold is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4 and the proof of the theorem is complete. □

4. The dimension of the (k, μ)-nullity distribution.

In the previous paragraph we considered the (k, μ)-nullity distribution $N(k, \mu)$ of the contact metric manifold $[M^{2n+1}, (\varphi, \xi, \eta, g)]$. Hence it is natural to ask how large $N(k, \mu)$ can be. If $k = \mu = 0$ then $R_{XY}\xi = 0$ for any X, Y and so the manifold locally is isometric to the product $E^{n+1}(0) \times S^n(4)$, with ξ belonging to the Euclidean factor [3]. Thus $\dim N(0,0) = n + 1$.

Recently, the following theorem has been proved [4]:

Theorem 4.1. *Let M^{2n+1} be a contact metric manifold with ξ belonging to the (k, μ)-nullity distribution. Then $k \leq 1$, and if $k = 1$ holds, then M is a Sasakian. If $k < 1$ then M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$. Moreover,*

$$(4.1)\quad \begin{aligned}
&1.\ R(X_\lambda, Y_\lambda)Z_{-\lambda} = (k-\mu)[g(\varphi X_\lambda, Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda, Z_{-\lambda})\varphi Y_\lambda]\\
&2.\ R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = (k-\mu)[g(\varphi Y_{-\lambda}, Z_\lambda)\varphi X_{-\lambda} - g(\varphi X_{-\lambda}, Z_\lambda)\varphi Y_{-\lambda}]\\
&3.\ R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = kg(\varphi X_\lambda, Z_{-\lambda})\varphi Y_{-\lambda} + \mu g(\varphi X_\lambda, Y_{-\lambda})\varphi Z_{-\lambda}\\
&4.\ R(X_\lambda, Y_{-\lambda})Z_\lambda = -kg(\varphi Y_{-\lambda}, Z_\lambda)\varphi X_\lambda - \mu g(\varphi Y_{-\lambda}, X_\lambda)\varphi Z_\lambda\\
&5.\ R(X_\lambda, Y_\lambda)Z_\lambda = [2(1+\lambda) - \mu][g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda]\\
&6.\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = [2(1-\lambda) - \mu][g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]
\end{aligned}$$

where $X_\lambda, Y_\lambda, Z_\lambda \in D(\lambda)$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in D(-\lambda)$.

We now state and prove the main result of this section.

Theorem 4.2. *Let $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a contact metric manifold of dimension $2n + 1 \geq 5$ such that ξ belongs to the (k, μ)-nullity distribution $N(k, \mu)$. If $k < 1$ and $k \neq 0$ then $\dim N(k, \mu) = 1$ and $N(k, \mu)$ is just the span of ξ.*

Proof: If $P \in M$ then by definition

$$(4.2)\quad \begin{aligned} N_P(k,\mu) = \{Z \in T_PM \mid R(X,Y)Z &= k(g(Y,Z)X - g(X,Z)Y)\\ &+ \mu(g(Y,Z)hX - g(X,Z)hY)\}. \end{aligned}$$

Suppose that there exist a unit vector $Z \in N(k, \mu)$ orthogonal to ξ. Then $Z = aZ_\lambda + bZ_{-\lambda}$ where $Z_\lambda, Z_{-\lambda}$ are unit vectors and $a, b \geq 0$.

Suppose that $X, Y \in D(\lambda)$, then using Theorem 4.1 we get

$$(4.3)\quad \begin{aligned} R(X,Y)Z = a&[2(1+\lambda) - \mu][g(Y, Z_\lambda)X - g(X, Z_\lambda)Y]\\ &+ b(k-\mu)[g(\varphi Y, Z_{-\lambda})\varphi X - g(\varphi X, Z_{-\lambda})\varphi Y]. \end{aligned}$$

On the other hand, from (4.2) we have

$$(4.4)\quad R(X,Y)Z = a(k + \lambda\mu)[g(Y, Z_\lambda)X - g(X, Z_\lambda)Y].$$

Now comparing these two equations, we get

$$(4.5)\quad \begin{aligned} a(1+\lambda)(1+\lambda-\mu)&[g(Y, Z_\lambda)X - g(X, Z_\lambda)Y]\\ &+ b(k-\mu)[g(\varphi Y, Z_{-\lambda})\varphi X - g(\varphi X, Z_{-\lambda})\varphi Y] = 0 \end{aligned}$$

for all $X, Y \in D(\lambda)$.

Suppose that $g(X,Y)=0$ and choose $\varphi Y = Z_{-\lambda}$. Then this equation is reduced to

$$a(1+\lambda)(1+\lambda-\mu)[g(Y,Z_\lambda)X - g(X,Z_\lambda)Y] = b(k-\mu)\cdot\varphi X = 0,$$

from which, by taking inner products with φX we deduce

$$b(k-\mu)=0 \tag{4.6}$$

and

$$a(1+\lambda)(1+\lambda-\mu)=0. \tag{4.7}$$

Now suppose that $X, Y \in D(-\lambda)$, then working similarly we get

$$\begin{aligned} b(\lambda-1)(\lambda+\mu-1)[g(Y,Z_{-\lambda})X - g(X,Z_{-\lambda})Y]& \\ + a(k-\mu)[g(\varphi Y,Z_\lambda)\varphi X - g(\varphi X,Z_\lambda)\varphi Y] &= 0. \end{aligned} \tag{4.8}$$

If we choose X, Y to be such that $g(X,Y)=0$ and $\varphi Y = Z_\lambda$ then the equation (4.8) is reduced to

$$b(\lambda-1)(\lambda+\mu-1)[g(Y,Z_{-\lambda})X - g(X,Z_{-\lambda})Y] + a(k-\mu)\varphi X = 0, \tag{4.9}$$

from which, taking the inner products with φX, we conclude that

$$a(k-\mu)=0 \tag{4.10}$$

and

$$b(\lambda-1)(\lambda+\mu-1)=0. \tag{4.11}$$

Now if $k \neq \mu$, (4.6) and (4.10) imply $a=b=0$ and the proof is complete, since we have $Z=0$. So suppose $k=\mu$. Then since $k = 1-\lambda^2$, (4.7) and (4.11) become

$$a\lambda(1+\lambda^2)=0 \tag{4.12}$$

and

$$b\lambda(\lambda-1)^2=0. \tag{4.13}$$

But $\lambda \neq 0$ $(k<1)$ and $\lambda \neq \pm 1$ $(k \neq 0)$ so we also conclude that $a=b=0$. Therefore, there does not exist a vector Z perpendicular to ξ belonging to the (k,μ)-nullity distribution, $N(k,\mu)$ is spanned by ξ and hence $\dim N(k,\mu)=1$. □

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