# Totally bounded frame quasi-uniformities

P. Fletcher, W. Hunsaker, W. Lindgren

*Abstract.* This paper considers totally bounded quasi-uniformities and quasi-proximities for frames and shows that for a given quasi-proximity $\lhd$ on a frame $L$ there is a totally bounded quasi-uniformity on $L$ that is the coarsest quasi-uniformity, and the only totally bounded quasi-uniformity, that determines $\lhd$. The constructions due to B. Banaschewski and A. Pultr of the Cauchy spectrum $\psi L$ and the compactification $\Re L$ of a uniform frame $(L, \mathbf{U})$ are meaningful for quasi-uniform frames. If $\mathbf{U}$ is a totally bounded quasi-uniformity on a frame $L$, there is a totally bounded quasi-uniformity $\overline{\mathbf{U}}$ on $\Re L$ such that $(\Re L, \overline{\mathbf{U}})$ is a compactification of $(L, \mathbf{U})$. Moreover, the Cauchy spectrum of the uniform frame $(Fr(\mathbf{U}^*), \mathbf{U}^*)$ can be viewed as the spectrum of the bicompletion of $(L, \mathbf{U})$.

*Keywords:* frame, uniform frame, quasi-uniform frame, quasi-proximity, totally bounded quasi-uniformity, uniformly regular ideal, compactification, bicompletion

*Classification:* 6D20, 18B35, 54D35, 54E05, 54E15

## 0. Introduction.

The concept of a quasi-proximity for a topological space was introduced by C.H. Dowker [4]. In [12] W. Hunsaker and W. Lindgren proved that there is a one-to-one correspondence between quasi-proximities and totally bounded quasi-uniformities and that each quasi-proximity class of quasi-uniformities contains a coarsest member, which is totally bounded. In this paper, we introduce the concept of a frame quasi-proximity, obtain results for frames analogous to those obtained for spaces in [12], and discuss compactifications of totally bounded quasi-uniform frames.

Let $\mathbf{U}$ be a totally bounded quasi-uniformity and let $L$ be the frame determined by $\mathbf{U}^*$. In [3] B. Banaschewski and A. Pultr give a compactification $\Re L$ of the uniform frame $(L, \mathbf{U}^*)$. We show that there exists a totally bounded quasi-uniformity $\overline{\mathbf{U}}$ on $\Re L$ such that $\overline{\mathbf{U}}^*$ determines $\Re L$ and that there exists a dense quasi-uniform frame homomorphism from $(\Re L, \overline{\mathbf{U}})$ onto $(L, \mathbf{U})$.

In the last section we consider briefly another construction from [3], the Cauchy spectrum of a uniform frame. We show that if $\mathbf{U}$ is a quasi-uniformity then the Cauchy spectrum of the underlying uniform frame $(Fr(\mathbf{U}^*), \mathbf{U}^*)$ can be constructed directly from the quasi-uniformity $\mathbf{U}$ in a manner that parallels the construction of the bicompletion of a quasi-uniform space [9].

## 1. Preliminaries.

A *frame* $(L, \leq)$ is a complete lattice that satisfies the frame distributive law: $a \wedge \bigvee S = \bigvee a \wedge x$ $(x \in S)$ for any $a \in L$ and any $S \subseteq L$. A function $f : L \to M$

between frames is a *join homomorphism* provided that for any $S \subseteq L, f(\bigvee S) = \bigvee\{f(s) : s \in S\}$. A join homomorphism that also preserves finite meets is called a *frame homomorphism*. We use 1 to denote $\bigwedge \varnothing$ and 0 to denote $\bigvee \varnothing$. A subset $C$ of a frame $(L, \leq)$ is a *cover* provided that $\bigvee C = 1$. For each $a \in L$, $\overline{a}$ denotes $\bigvee\{x \in L : x \wedge a = 0\}$; this element $\overline{a}$ is called the *pseudocomplement* of $a$. Throughout this paper if $F$ is a collection of functions mapping a frame $L$ to a frame $M$ we define $\bigwedge F$ pointwise and for $u, v \in F$ we write $u \leq v$ to mean that for each $x \in L$, $u(x) \leq v(x)$.

We recall the following fundamental concepts and results from [8].

For $a$ and $b$ in $L$, the function $a \sharp b : L \to L$ is defined by

$$a \sharp b(x) = \begin{cases} b & \text{if } a \wedge x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $u : L \to L$ is any function and $x \in L$, then $x$ is *$u$-small* provided that $x \sharp x \leq u$. The collection of all $u$-small elements is denoted by $S_u$, and if $u$ is an order-preserving function such that $\bigvee S_u = 1$ we say that $u$ is a *$\Delta$-map*.

A *frame quasi-uniformity base* supported on a frame $(L, \leq)$ is a collection $\mathbf{B}$ of $\Delta$-maps such that

(1) For each $u \in \mathbf{B}$ there exists $v \in \mathbf{B}$ such that $v \circ v \leq u$.
(2) For $u, v \in \mathbf{B}$ there is a join homomorphism $w$ and a $z \in \mathbf{B}$ such that $z \leq w \leq u \wedge v$.

If $\mathbf{B}$ is a frame quasi-uniformity base, then the quasi-uniformity $\mathbf{U}$ for which $\mathbf{B}$ is a base is the collection of all $w : L \to L$ such that $w$ is order preserving and there is a $u \in \mathbf{B}$ with $u \leq w$. The members of a quasi-uniformity $\mathbf{U}$ are called *entourages*.

If $\mathbf{B}$ satisfies:

(3) For each $u \in \mathbf{B}$ and for each $x, y \in L, u(x) \wedge y = 0$ if and only if $u(y) \wedge x = 0$, then $\mathbf{B}$ is a *base for a frame uniformity for $L$*.

A collection $\mathbf{D}$ of $\Delta$-maps is a *subbase* for a frame quasi-uniformity $\mathbf{U}$ provided that the collection of all finite meets from $\mathbf{D}$ is a base for $\mathbf{U}$.

The frame of $\mathbf{U}$, denoted by $Fr(\mathbf{U})$ is the collection to which $a$ belongs provided that

$$a = \bigvee\{b \in L : u(b) \leq a \quad \text{for some} \quad u \in \mathbf{U}\}.$$

We say that $\mathbf{U}$ *determines* $L$ provided that $Fr(\mathbf{U}) = L$.

Let $\mathbf{U}$ and $\mathbf{V}$ be quasi-uniformities on frames $L$ and $M$ respectively and let $f : L \to M$ be a frame homomorphism. Then $f$ is a *quasi-uniform frame homomorphism* provided that for every $u \in \mathbf{U}$ there exists a $v \in \mathbf{V}$ such that $v \circ f \leq f \circ u$.

For each $\Delta$-map $u$ and each $x \in L$ define

$$\widehat{u} : L \to L \quad \text{by} \quad \widehat{u}(x) = \bigvee\{b \sharp a : a \sharp b \leq u\}(x)$$

and

$$u^* : L \to L \quad \text{by} \quad u^*(x) = \bigvee\{a : a \sharp a \leq u \text{ and } a \wedge x \neq 0\}.$$

Then for any quasi-uniformity $\mathbf{U}$ supported on a frame $L$, $\{\hat{u} : u \in \mathbf{U}\}$ is a base for a quasi-uniformity $\hat{\mathbf{U}}$ on $L$ and $\{u^* : u \in \mathbf{U}\}$ is a base for a uniformity $\mathbf{U}^*$ on $L$ that is the coarsest quasi-uniformity containing $\mathbf{U} \cup \hat{\mathbf{U}}$. The *underlying biframe* of $\mathbf{U}$ is the triple $(Fr(\mathbf{U}^*), Fr(\mathbf{U}), Fr(\hat{\mathbf{U}}))$. It is shown in [8] that the underlying biframe of $\mathbf{U}$ is a biframe in the sense of B. Banaschewski, G.C.L. Brümmer and K. Hardie [2]. If $\mathbf{U}$ is a quasi-uniformity on $L$ and $\mathbf{U}^*$ determines $L$, we say that $(L, \mathbf{U})$ is a *quasi-uniform frame.*

## 2. Quasi-proximities.

In this section we extend the theory of quasi-proximities established in [12] to a theory of quasi-proximities for frames.

**Definition.** Let $(L, \leq)$ be a frame. A *quasi-proximity on* $L$ is a binary relation $\triangleleft$ on $L$ satisfying the following axioms for $a$, $b$, $c$, $d$ in $L$.

(1) $0 \triangleleft 0$ and $1 \triangleleft 1$.
(2) If $a \triangleleft b$, then $a \leq b$.
(3) If $a \leq b \triangleleft c \leq d$, then $a \triangleleft d$.
(4) If $a \triangleleft b$ and $a \triangleleft c$, then $a \triangleleft b \wedge c$.
(5) If $a \triangleleft c$ and $b \triangleleft c$, then $a \vee b \triangleleft c$.
(6) If $a \triangleleft b$, then there exists $c \in L$ such that $a \triangleleft c \triangleleft b$.
(7) If $a \triangleleft b$, then $\overline{a} \vee b = 1$.

**Proposition 2.1.** *Let $(L, \leq)$ be a frame and let $\mathbf{U}$ be a quasi-uniform base on $L$. For $a, b \in L$ define $a \triangleleft b$ if and only if $u(a) \leq b$ for some $u \in \mathbf{U}$. Then $\triangleleft$ is a quasi-proximity on $L$.*

Proof: The axioms (1) – (5) follow easily from the properties of a quasi-uniformity and axiom (6) holds as in the proof of [8, Proposition 5.1]. To see that axiom (7) holds suppose that $a \triangleleft b$ and let $u \in \mathbf{U}$ such that $u(a) \leq b$. It suffices to show that $\overline{a} \vee u(a) = 1$. We have $1 = \bigvee\{x \in L : x \text{ is } u\text{-small}\} = \bigvee\{x \in L : x \text{ is } u\text{-small and } x \wedge a \neq 0\} \vee \bigvee\{x \in L : x \text{ is } u\text{-small and } x \wedge a = 0\} \leq u(a) \vee \overline{a}$. □

**Definition.** If $\mathbf{U}$ is a quasi-uniformity (base) on a frame $L$, then the quasi-proximity $\triangleleft$ defined by $a \triangleleft b$ if and only if $u(a) \leq b$ for some $u \in \mathbf{U}$ is called the *quasi-proximity determined by* $\mathbf{U}$.

**Lemma 2.2.** *Let $(L, \leq)$ be a frame. Let $C = \{(a_\alpha, b_\alpha) : a_\alpha, b_\alpha \in L, \alpha \in A\}$ and suppose that for each $B \subseteq A$, $(\bigwedge_{\alpha\in B} a_\alpha, \bigwedge_{\alpha\in B} b_\alpha) \in C$ and $(\bigvee_{\alpha\in B} a_\alpha, \bigvee_{\alpha\in B} b_\alpha) \in C$. For each $\alpha \in A$ and each $x \in L$, let*

$$u_\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ b_\alpha & \text{if } x \leq a_\alpha \text{ and } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

*and let $u(x) = \bigwedge u_\alpha(x)$. Then $u : L \to L$ is a join homomorphism.*

Proof: Let $x = \bigvee x_i$. Then for each $\alpha \in A$ and each $i$, $u_\alpha(x) \geq u_\alpha(x_i)$ and so $u(x) \geq \bigvee_i u(x_i)$. In order to show that $u(x) \leq \bigvee_i u(x_i)$ we may suppose that for each $i$, $u(x_i) \neq 1$ and for some $i, u(x_i) \neq 0$. For each $i$, let $B_i = \{\alpha : x_i \leq a_\alpha\}$. Then $B_i \neq \emptyset$. Let $w_i = \bigwedge\{a_\alpha : \alpha \in B_i\}$, $z_i = \bigwedge\{b_\alpha : \alpha \in B_i\}$. Then for each $i$, $(w_i, z_i) \in C$, $x_i \leq w_i$ and $u(x_i) = z_i$. Let $w = \bigvee w_i$ and let $z = \bigvee z_i$. Then $(w, z) \in C$; hence $(w, z) = (a_\gamma, b_\gamma)$ for some $\gamma \in A$ and $u(x) \leq u_\gamma(x) = z = \bigvee_i u(x_i)$.

□

**Definition.** Let $L$ be a frame and let $\mathbf{U}$ be a quasi-uniformity on $L$. Then $\mathbf{U}$ is *totally bounded* provided that for each $u \in \mathbf{U}$ there is a finite cover of $L$ by $u$-small elements.

**Theorem 2.3.** *Let $L$ be a frame and let $\lhd$ be a quasi-proximity on $L$. For $a, b \in L$ define*

$$u_{a,b}(x) = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } x \leq a, x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

*and let $\mathcal{S} = \{u_{a,b} : a \lhd b\}$. Then $\mathcal{S}$ is a subbase for a totally bounded frame quasi-uniformity $\mathbf{U}_\lhd$, which determines $\lhd$, and is the only totally bounded frame quasi-uniformity that determines $\lhd$.*

Proof: We first prove that $\mathcal{S}$ is a subbase for a quasi-uniformity. Let $a, b \in L$ and suppose that $a \lhd b$. Then $\overline{a}$ and $b$ are $u_{a,b}$-small and so $u_{a,b}$ is a $\Delta$-map. Let $u_{a_i,b_i} \in \mathcal{S}$, $1 \leq i \leq n$. Let $D = \{(a_i, b_i) : 1 \leq i \leq n\}$ and form $C = \{(a_\alpha, b_\alpha) : \alpha \in A\}$ by taking all meets and joins from $D$. Let $u = \bigwedge_{\alpha \in A} u_\alpha$ and note that $u \leq \bigwedge_{i=1}^{n} u_{a_i,b_i}$. It follows from Lemma 2.2 that $u$ is a join homomorphism that is a finite meet of members of $\mathcal{S}$. Moreover, $u$ is a $\Delta$-map.

Let $u_{a,b} \in \mathcal{S}$. There exists $c \in L$ such that $a \lhd c \lhd b$. Let $w = u_{a,c} \wedge u_{c,b}$. It is easy to verify that $w^2 \leq u_{a,b}$. Therefore $\mathcal{S}$ is a subbase for a frame quasi-uniformity $\mathbf{U}_\lhd$. If $u_{a,b} \in \mathcal{S}$, then $\{\overline{a}, b\}$ is a cover of $L$ by $u_{a,b}$-small elements. It follows that $\mathbf{U}_\lhd$ is totally bounded.

We now show that $\mathbf{U}_\lhd$ determines $\lhd$. Let $\lhd_1$ denote the quasi-proximity determined by $\mathbf{U}_\lhd$. Suppose that $a \lhd b$. Then $u_{a,b}(a) \leq b$ and hence $a \lhd_1 b$. Now suppose that $a \lhd_1 b$. There exists $u \in \mathbf{U}_\lhd$ such that $u(a) \leq b$. Since $u \in \mathbf{U}_\lhd$, there are $(a_i, b_i)$, $1 \leq i \leq n$, such that $a_i \lhd b_i$ for each $i$, and $\bigwedge_{i=1}^{n} u_{a_i,b_i} \leq u$. Let $w = \bigwedge_{i=1}^{n} u_{a_i,b_i}$. Let $J = \{i : a \leq a_i\}$, and let $c = \bigwedge_{j \in J} a_j$, $d = \bigwedge_{j \in J} b_j$. Then $a \leq c \lhd d \leq b$.

We next show that $\mathbf{U}_\lhd$ is the coarsest frame quasi-uniformity that determines $\lhd$. Suppose that $\mathbf{V}$ is a frame quasi-uniformity that determines $\lhd$. Let $u_{a,b} \in \mathcal{S}$; then $a \lhd b$ so there exists a join homomorphism $v \in \mathbf{V}$ such that $v(a) \leq b$. It follows that $v \leq u_{a,b}$.

Finally we show that $\mathbf{U}_\lhd$ is the only totally bounded frame quasi-uniformity that determines $\lhd$. Suppose that $\mathbf{V}$ is a totally bounded frame quasi-uniformity that determines $\lhd$. Let $w \in V$ and let $v \in V$ such that $v^2 \leq w$. There exists a finite cover $\{a_i\}$ of $L$ by $v$-small elements. Since $V$ determines $\lhd$, we have that $a_i \lhd v(a_i)$

for all $i$. Note that $u_{a_i,v(a_i)} \in \mathbf{U}_\triangleleft$ and let $z \in \mathbf{U}_\triangleleft$ be a join homomorphism such that $z \leq \bigwedge_i u_{a_i,v(a_i)}$. To see that $z \leq w$ let $x \in L$. Then $z(x) = \bigvee_i z(x \wedge a_i)$. For each $j$,

$$\begin{aligned} z(x \wedge a_j) \leq & \bigwedge_i u_{a_i,v(a_i)}(x \wedge a_j) \\ \leq & u_{a_j,v(a_j)}(a_j) \leq v(a_j) \leq v^2(x) \leq w(x). \end{aligned}$$

□

## 3. Compactifications of totally bounded quasi-uniform frames.

Let $\mathbf{U}$ be a totally bounded quasi-uniformity and let $(L, L_1, L_2)$ be the underlying biframe of $\mathbf{U}$. Let $\triangleleft^*$ be the quasi-proximity determined by $\mathbf{U}^*$. We note that $\triangleleft^*$ is the "uniformly below" relation of [3, p. 63]. For the remainder of this paper we follow the notation and terminology of [3] and make use of the results contained therein. In particular, an ideal $J$ in $L$ is *uniformly regular* provided that if $x \in J$ there is a $y \in J$ with $x \triangleleft^* y$; $\Re L$ denotes the frame of all uniformly regular ideals of $L$ and $k(x)$ is the uniformly regular ideal consisting of all $y \in L$ such that $y \triangleleft^* x$. In [3] the authors establish that $\Re L$ is a compactification of the uniform frame $(L, \mathbf{U}^*)$. The purpose of this section is to show that there exists a totally bounded quasi-uniformity $\overline{\mathbf{U}}$ on $\Re L$ such that $\overline{\mathbf{U}}^*$ determines $\Re L$ and a dense quasi-uniform frame homomorphism from $(\Re L, \overline{\mathbf{U}})$ onto $(L, \mathbf{U})$. That is, we show that $(\Re L, \overline{\mathbf{U}})$ is a compactification of the quasi-uniform frame $(L, \mathbf{U})$.

For each $u \in \mathbf{U}$ define $\overline{u} : \Re L \to \Re L$ by $\overline{u}(J) = \bigvee\{k(u(x)) : x \in S_u$ and $x \wedge \bigvee J \neq 0\}$, and let $\overline{\mathbf{B}} = \{\overline{u} : u \in \mathbf{U}\}$. We show that $\overline{\mathbf{B}}$ is a base for a quasi-uniformity $\overline{\mathbf{U}}$ supported on $\Re L$ such that $\overline{\mathbf{U}}^*$ determines $\Re L$, and such that $(\Re L, \overline{\mathbf{U}})$ is a compactification of the quasi-uniform frame $(L, \mathbf{U})$.

In order to establish that $(\Re L, \overline{\mathbf{U}})$ is a quasi-uniform frame, we need the following lemmas.

**Lemma 3.1.** *Let $u \in \mathbf{U}$. If $x$ is a $u$-small element of $L$, then $k(x)$ is $\overline{u}$-small, and if $J \in \Re L$ is $\overline{u}$-small and $x \in J$, then $x$ is $u^2$-small.*

Proof: Let $x$ be a $u$-small element of $L$. Let $J \in \Re L$ such that $J \cap k(x) \neq \{0\}$. Let $y \in k(x)$ and let $a \in J \cap k(x)$, $a \neq 0$. Then $a \wedge x \neq 0$ and so $x \leq u(a)$. Thus $y \triangleleft^* x \leq u(a)$ and so $y \in k(u(a))$. Therefore $k(x) \subseteq k(u(a)) \subseteq \overline{u}(J)$.

Let $J$ be a $\overline{u}$-small element of $\Re L$ and let $x \in J$. Suppose that $y \wedge x \neq 0$. Since $x \in J$, $k(x) \subseteq J$ and since $J$ is $\overline{u}$-small, $k(x)$ is $\overline{u}$-small. Note that $k(x \wedge y) \subseteq k(x) \wedge k(y)$ and $0 \neq x \wedge y = \bigvee k(x \wedge y)$ so that $k(x) \leq \overline{u}(k(y))$. Thus $x = \bigvee k(x) \leq \bigvee \overline{u}(k(y))$. Let $a \in \overline{u}(k(y))$. Then $a = \bigvee_{i=1}^{n} a_i$ where for $1 \leq i \leq n$ there exist $z_i$ and $q_i$ such that $a_i \triangleleft^* u(z_i)$, $z_i$ is $u$-small, $z_i \wedge q_i \neq 0$ and $q_i \triangleleft^* y$. For $1 \leq i \leq n$, $z_i \leq u(q_i) \leq u(y)$ and so $a_i \leq u(z_i) \leq u^2(y)$. Hence $x \leq \bigvee \overline{u}(k(y)) \leq u^2(y)$.

**Lemma 3.2.** *Let $a, b \in L$ and suppose that $u \in \mathbf{U}$ such that $u^*(b) \leq a$. Let $w \in \mathbf{U}$ such that $w^4 \leq u$. Then $\overline{w}^*(k(b)) \subseteq k(a)$.*

Proof: Let $J$ be a $\overline{w}$-small member of $\Re L$ such that $J \cap k(b) \neq \{0\}$. Let $y \in J$ and $z \in J \cap k(b)$, $z \neq 0$. Then $y \vee z \in J$ and by Lemma 3.1, $y \vee z$ is $w^2$-small. Therefore by [8, Proposition 2.1], $y \leq y \vee z \leq (w^2)^*(b) \lhd^* u^*(b) \leq a$ and so $J \subseteq k(a)$. □

**Proposition 3.3.** *Let* $\mathbf{U}$ *be a totally bounded frame quasi-uniformity and let* $L = Fr(\mathbf{U}^*)$. *Let* $\overline{\mathbf{B}} = \{\overline{u} : u \in \mathbf{U}\}$. *Then* $\overline{\mathbf{B}}$ *is a base for a totally bounded frame quasi-uniformity* $\mathbf{U}$ *such that* $(\Re L, \overline{\mathbf{U}})$ *is a quasi-uniform frame.*

Proof: Let $u \in \mathbf{U}$, let $J \in \Re L$ and let $a \in J$. Since $\mathbf{U}$ is totally bounded, $a = \bigvee_{i=1}^{n} a_i$ where each $a_i \in S_u$. Thus $a = \bigvee_{i=1}^{n} a_i \in \bigvee_{i=1}^{n} k(u(a_i)) \subseteq \overline{u}(J)$. Hence $J \subseteq \overline{u}(J)$ and it is clear that $\overline{u}$ is a join homomorphism.

Let $w \in \mathbf{U}$ and let $u \in \mathbf{U}$ such that $u^3 \leq w$, and let $J \in \Re L$.

$$\begin{aligned}\overline{u}(\overline{u}(J)) &= \overline{u}\Big(\bigvee\Big\{k(u(c)) : c \in S_u \text{ and } c \wedge \bigvee J \neq 0\Big\}\Big)\\ &= \bigvee\{\overline{u}(k(u(c))) : c \in S_u \text{ and } c \wedge \bigvee J \neq 0\}\\ &= \bigvee\{k(u(b)) : b, c \in S_u, b \wedge \bigvee k(u(c)) \neq 0, \text{ and } c \wedge \bigvee J \neq 0\}\\ &\subseteq \bigvee\{k(w(c)) : c \in S_w \text{ and } c \wedge \bigvee J \neq 0\}\\ &= \overline{w}(J).\end{aligned}$$

To see that axiom (2) holds for $\overline{\mathbf{B}}$, let $u, w \in \mathbf{U}$ and let $J \in \Re L$.

$$\begin{aligned}&\overline{u \wedge w}(J) = \bigvee\{k((u \wedge w)(a)) : a \in S_{u\wedge w} \text{ and } a \wedge \bigvee J \neq 0\}\\ &\subseteq \bigvee\{k(u(b) \wedge w(c)) : b, c \in S_{u\wedge w}, b \wedge \bigvee J \neq 0, \text{ and } c \wedge \bigvee J \neq 0\}\\ &\subseteq \bigvee\{k(u(b)) : b \in S_u \text{ and } b \wedge \bigvee J \neq 0\} \cap \bigvee\{k(w(c)) : c \in S_w \text{ and } c \wedge \bigvee J \neq 0\}\\ &= \overline{u}(J) \cap \overline{w}(J).\end{aligned}$$

Let $u \in \mathbf{U}$. Since $\mathbf{U}$ is totally bounded, there is a finite subcover $A$ of $S_u$. Banaschewski and Pultr [3, p. 67] prove that $\bigvee\{k(x) : x \in A\} = L$. Thus, it follows from Lemma 3.1 that for each $u \in \mathbf{U}$, $\overline{u}$ is a $\Delta$-map and it also follows that $\overline{\mathbf{U}}$ is totally bounded.

It remains to show that $\overline{\mathbf{U}}^*$ determines $\Re L$. Let $J \in \Re L$. Then $J = \bigvee\{k(a) : k(a) \subseteq J\}$. Let $b \in J$. There exists $a \in J$ such that $b \lhd^* a$. By Lemma 3.2, there exists $w \in \mathbf{U}$ such that $\overline{w}^*(k(b)) \subseteq k(a) \subseteq J$. Hence $k(b) \lhd^* J$. □

**Proposition 3.4.** *The function* $g : (\Re L, \overline{\mathbf{U}}) \to (L, \mathbf{U})$ *defined by join is a dense quasi-uniform frame homomorphism onto* $(L, \mathbf{U})$.

Proof: Let $a \in L$. Since $a = \bigvee\{b : b \lhd^* a\} = \bigvee k(a)$, $g$ maps onto $(L, \mathbf{U})$. Clearly $g^{-1}(0) = \{0\}$. Let $\overline{u} \in \overline{\mathbf{U}}$ and let $v \in \mathbf{U}$ such that $v^2 \leq u$. We show that $v \circ g \leq g \circ \overline{u}$. Let $J \in \Re L$. Then $\overline{u}(J) = \bigvee\{k(u(a)) : a \in S_u \text{ and } a \wedge \bigvee J \neq 0\}$

and $g \circ \overline{u}(J) = \bigvee\big(\bigvee\{k(u(a)) : a \in S_u \text{ and } a \wedge \bigvee J \neq 0\}\big)$. On the other hand $v \circ g(J) = v(\bigvee J) = \bigvee\{v(a) : a \in S_u \text{ and } a \wedge \bigvee J \neq 0\}$. Since $v(a) \lhd^* u(a)$, $v(a) \in \bigvee\big(\bigvee\{k(u(a)) : a \in S_u \text{ and } a \wedge \bigvee J \neq 0\}\big)$.

It follows from Theorem 3.2 that $\overline{\mathbf{U}}^*$ is a uniformity that determines $\Re L$ and it follows from [3, Corollary to Lemma 2 and Lemma 4] that $\overline{\mathbf{U}}^*$ is the only uniformity that determines $\Re L$. The join map from $(\Re L, \overline{\mathbf{U}})$ to $(L, \mathbf{U})$ is the required dense quasi-uniform frame homomorphism. □

## 4. The bicompletion of a quasi-uniform frame.

In this final section, we consider the sense in which the Cauchy spectrum of a quasi-uniform frame, introduced by Banaschewski and Pultr [3], can be viewed as the spectrum of its bicompletion. We make use of the result [3, Proposition 9] that the Cauchy spectrum of a uniform frame $(L,\mathcal{U})$ is the spectrum of its completion $CL$. In order to make this section dovetail with [3], we use covering uniformities. For a given quasi-uniform frame $(L, \mathbf{U})$ the collection of covers $\{S_u : u \in \mathbf{U}\} = \{S_u : u \in \mathbf{U}^*\}$ generates the covering uniformity $\mathcal{U}$ corresponding to the entourage uniformity $\mathbf{U}^*$ [5]. Let $(L, \mathbf{U})$ be a quasi-uniform frame. A filter $F$ in $L$ is a $\mathbf{U}$-*Cauchy filter* provided that for each $u \in \mathbf{U}$, $S_u \cap F \neq \varnothing$. It is shown in [3] that a $\mathbf{U}^*$-Cauchy filter is $\mathbf{U}^*$-regular if, and only if, it is a minimal $\mathbf{U}^*$-Cauchy filter. Given a covering uniformity $\mathcal{U}$, Banaschewski and Pultr construct the uniform space $\psi L$ whose ground set is the collection of all minimal Cauchy filters and whose uniformity is generated by the covers $\psi_A = \{\psi_a : a \in A\}$ where $A \in \mathcal{U}$ and for each $a \in A, \psi_a = \{F \in \psi L : a \in F\}$. They call the resulting uniform space the *Cauchy spectrum* of the uniform frame $(L,\mathcal{U})$.

We make repeated use of the following proposition.

**Proposition 4.1** [8]**.** *Let $(X,\mathcal{U})$ be a quasi-uniform space, let $A$ and $B$ be $\mathcal{T}(\mathcal{U})$-open sets and let $U$ be an open neighbornet of $X$. Let $u : \mathcal{T}(\mathcal{U}) \to \mathcal{T}(\mathcal{U})$ be defined by $u(G) = U(G)$. If $A \times B \subseteq U$, then $A \sharp B \leq u$. If $A \sharp B \leq u$, then $A \times B \subseteq \overline{U}$, where the closure is taken either with respect to $\mathcal{T}(\mathcal{U}) \times \mathcal{T}(\mathcal{U})$ or with respect to $\mathcal{T}(\mathcal{U}) \times \mathcal{T}(\mathcal{U}^{-1})$.*

**Proposition 4.2.** *Let $\mathbf{U}$ be a frame quasi-uniformity and let $L = Fr(\mathbf{U}^*)$. For each $u \in \mathbf{U}$ set $\tilde{u} = \{(F,G) \in \psi L \times \psi L$: there exist $x \in F$ and $y \in G$ such that $x \sharp y \leq u\}$. Then $\tilde{\mathbf{U}} = \{\tilde{u} : u \in \mathbf{U}\}$ is a base for a quasi-uniformity on $\psi L$ and $(\psi L, \tilde{\mathbf{U}}^*)$ is the Cauchy spectrum of $L$.*

Proof: We first prove that $\tilde{\mathbf{U}}$ is a base for a quasi-uniformity on $\psi L$. Let $u, v \in \mathbf{U}$. Then $u \wedge v \in \mathbf{U}$ and $\widetilde{u \wedge v} = \tilde{u} \cap \tilde{v}$. Moreover, for each $F \in \psi L$ there exists a $u$-small $x \in F$ and since $x \sharp x \leq u$, $(F,F) \in \tilde{u}$.

Let $u \in \mathbf{U}$ and let $v \in \mathbf{U}$ such that $v^2 \leq u$. To show that $\tilde{v}^2 \subseteq \tilde{u}$, let $(F,G)$ and $(G,H)$ belong to $\tilde{v}$. There are $x$ in $F$ and $y \in G$ such that $x \sharp y \leq v$ and $p$ in $G$ and $q$ in $H$ such that $p \sharp q \leq v$. Since $y \wedge p \neq 0$, $x \sharp q \leq u$. Thus $\tilde{v}^2 \subseteq \tilde{u}$.

In view of [8, Proposition 2.1] and the introductory remarks of this section, in order to show that $(\psi L, \tilde{\mathbf{U}}^*)$ is the Cauchy spectrum it suffices to prove that $\{S_{\tilde{u}} : \tilde{u} \in \tilde{\mathbf{U}}\}$ is a base for the covering uniformity given by Banaschewski and

Pultr [3]. Let $w \in \mathbf{U}$ and let $z, v \in \mathbf{U}$ such that $v^3 \leq w$ and $z^2 \leq v$. There exists $\tilde{u} \in \tilde{\mathbf{U}}$ such that $\tilde{u}$ is closed in the topology $\tau(\tilde{\mathbf{U}}) \times \tau(\tilde{\mathbf{U}}^{-1})$ and $\tilde{u} \subseteq \tilde{z}$ [9, page 8]. We show that $S_{\tilde{u}}$ refines $\psi_{S_w}$. Let $T \in S_{\tilde{u}}$. Since $T$ is a $\tilde{u}$-small set of minimal $\mathbf{U}$-Cauchy filters, $T \sharp T \leq \tilde{u}$ and by Proposition 4.1, $T \times T \subseteq \tilde{u}$. Let $F \in T$ and let $a \in F \cap S_z$. We show that $T \subseteq \psi_{v^*(a)}$. Let $G \in T$. There exist $x_1, x_2 \in F$ and $y_1, y_2 \in G$ such that $x_1 \sharp y_1 \leq z$ and $y_2 \sharp x_2 \leq z$. Set $x = x_1 \wedge x_2$ and $y = y_1 \wedge y_2$ and note that $x \neq 0$, $y \neq 0$, $y \in G$, $x \sharp y \leq z$ and $y \sharp x \leq z$. By definition, $x \sharp y \leq \widehat{z}$ and so $y \leq z(a) \wedge \widehat{z}(a)$. It follows from [8, Lemma 3.12] that $y \leq v^*(a)$ and so $G \in \psi_{v^*(a)}$. By [8, Proposition 3.9 (2)], $v^*(a)$ is $v^3$-small; hence $\psi_{v^*(a)} \in \psi_{A_w}$. Thus $S_{\tilde{u}}$ refines $\psi_{S_w}$.

To show that $\psi_{S_u} \subseteq S_{\tilde{u}}$, let $a \in S_u$ and let $F, G, \in \psi_a$. Then $a \in F \cap G$ and $a \sharp a \leq u$ so that $(F, G) \in \tilde{u}$. Then $\psi_a \times \psi_a \subseteq \tilde{u}$ and so by Proposition 4.1, $\psi_a \in S_{\tilde{u}}$. □

It follows from Proposition 4.1 and the proof of [9, Theorem 3.33] that $(\psi L, \mathbf{U}^*)$ is the bicompletion of $(L, \mathbf{U})$ whenever $\mathbf{U}$ is a quasi-uniformity on a set $X$ and $L = \mathcal{T}(\mathbf{U}^*)$.

## References


[1] Banaschewski B., Brümmer G.C.L., *Strong zero-dimensionality of biframes and bispaces*, Quaestiones Math. **13** (1990), 273–290.

[2] Banaschewski B., Brümmer G.C.L., Hardie K.A., *Biframes and bispaces*, Quaestiones Math. **6** (1983), 13–25.

[3] Banaschewski B., Pultr A., *Samuel compactification and completion of uniform frames*, Math. Proc. Camb. Phil. Soc. (1) **108** (1990), 63–78.

[4] Dowker C.H., *Mappings of proximity structures*, "General Topology and its Relations to Modern Analysis and Algebra", (Proc. Sympos. Prague 1961), Academic Press, New York, 1962, 139–141; Publ. House Czech. Acad. Sci., Prague, 1962.

[5] Fletcher P., Hunsaker W., *Entourage uniformities for frames*, Monatsh. Math. **112** (1991), 271–279.

[6] ———, *Symmetry conditions in terms of open sets*, Topology and its Appl. **45** (1992), 39–47.

[7] Fletcher P., Hunsaker W., Lindgren W., *Characterizations of Frame Quasi-Uniformities*, preprint.

[8] ———, *Frame Quasi-Uniformities*, preprint.

[9] Fletcher P., Lindgren W., *Quasi-Uniform Spaces*, Marcel Dekker, New York and Basel, 1982.

[10] Frith J.L., *Structured Frames*, Ph.D. Thesis, Univ. Cape Town, 1987.

[11] Gantner T.E., Steinlage R.C., *Characterizations of quasi-uniformities*, J. London Math. Soc. (2) **5** (1972), 48–52.

[12] Hunsaker W., Lindgren W.F., *Construction of quasi-uniformities*, Math. Ann. **188** (1970), 39–42.

[13] Isbell J.R., *Uniform Spaces*, Mathematical Surveys, No. 12, Amer. Math. Soc., Providence, R.I., 1964.

[14] Johnstone P.T., *Stone Spaces*, Cambridge Univ. Press, Cambridge, 1982.

[15] Nachbin L., *Sur les espaces uniformes ordonnés*, C.R. Acad. Sci., Paris **226** (1948), 774–775.
[16] Pultr A., *Pointless uniformities I. Complete regularity*, Comment Math. Univ. Carolinae **25** (1984), 91–104.

Mathematics Dept., Virginia Polytechnic Institute, Blacksburg, VA 24061, USA

Mathematics Dept., Southern Illinois University, Carbondale, IL 62901, USA

Mathematics Dept., Slippery Rock University, Slippery Rock, PA 16057, USA

(*Received* November 23, 1992)