A finite dimensional reduction of the Schauder Conjecture

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Abstract. Schauder's Conjecture (i.e. every compact convex set in a Hausdorff topological vector space has the f.p.p.) is reduced to the search for fixed points of suitable multivalued maps in finite dimensional spaces.

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Classification: 47H10, 46A16, 54C60

1. Throughout the paper E denotes a real Hausdorff topological vector space, D a compact convex subset of E, $f : D \to D$ a continuous map.

If E is locally convex, f has a fixed point after the Tychonoff theorem [21]. If E is a metric linear space (not necessarily locally convex), Schauder [20] proved that f has a fixed point. Unfortunately the proof is not correct. The existence of a fixed point for f, without additional assumptions about E, is known as the Schauder Conjecture (Problem 54 in the Scottish Book [16]).

The Schauder Conjecture is still open even in the metric case.

The aim of this paper is to show that the Schauder Conjecture would be positively solved, if we were able to find fixed points for suitable multivalued maps in finite dimensional spaces.

During the last years the Schauder Conjecture has been intensively examined by many mathematicians. It is not the occasion to cite the many valuable contributions, especially from the Schools of Aachen, Dresden and Novi Sad [12], to our understanding of the problem.

However, in the concluding section, we give some information about the state of the Schauder Conjecture, not usually well known, as far as we are aware, to the fixed-point people.

2. Let $\mathcal{T}(0)$ be a fundamental system of neighbourhoods of 0 in E. Without loss of generality we can suppose that every $V \in \mathcal{T}(0)$ is closed and star-shaped with respect to 0.

Fixed $V \in \mathcal{T}(0)$, since D is compact, there exist $x_1, \dots, x_n \in D$ such that $D \subseteq \bigcup_{i=1}^{n} V_i$, where $V_i = (V + x_i) \cap D$. Since D is convex and $V + x_i$ is star-shaped with respect to x_i, such is V_i as well.

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The following sets play a relevant role in the sequel:

$$I = \{1, 2, \dots, n\},\ \alpha = \left\{ L : L \subseteq I,\ \bigcap_{i \in L} V_i \neq \phi \right\},$$
$$\beta = \{L \in \alpha,\ L \text{ maximal with respect to the set inclusion.}\}$$

For every $L \in \beta$ choice a point $y_L \in \bigcap_{i \in L} V_i$, and denote by P the convex hull of the finite set $\{x_i, i \in I\} \cup \{y_L, L \in \beta\}$.

For every $x \in P$ denote by $L(x)$ the set $\{i \in I, f(x) \in V_i\}$. Obviously $L(x)$ in non empty and $L(x) \in \alpha$.

Now we are in a position to define a multivalued map $\varphi : P \to P$ in the following way:

$$\varphi(x) = \cup \{[x_i, y_L], i \in L(x) \text{ and } L(x) \subseteq L \in \beta\},$$

where $[x_i, y_L]$ is the closed segment joining x_i with y_L.

We want to illustrate some properties of φ.

(a) For every $x \in P$

$$\varphi(x) \subseteq \cup \{V_i, i \in L(x)\}.$$

In fact if $L(x) \subseteq L \in \beta$, we have

$$y_L \in \cap \{V_i, i \in L\} \subseteq \cap \{V_i, i \in L(x)\}$$

and $[x_i, y_L] \subseteq V_i$ since V_i is star-shaped with respect to x_i.

(b) $\varphi(x)$ is union of closed segments; each of these segments joins a point x_i $(i \in L(x))$ with a point $y_L (L(x) \subseteq L \in \beta)$:
$\varphi(x)$ is a connected compact one-dimensional set.

Nevertheless the homologic structure of $\varphi(x)$ may be very complicated: in fact $\varphi(x)$ can have a high number of cycles.

(c) φ has only a finite number of values. Consequently the search for fixed points of φ is combinatoric in its own nature. In some sense this fact belongs to the deep nature of fixed point theory.

(d) The main reason to consider the map φ is that φ enjoys a strengthened form of upper semicontinuity [1]: if $z_n \in P$ and $z_n \to z$ then $\varphi(z_n) \subseteq \varphi(z)$ definitively.

Indeed by the definition of $L(z)$ we have $f(z) \notin V_j$, for every $j \in I \setminus L(z)$.

Since E is a Hausdorff space there exists a neighbourhood U of $f(z)$ such that $U \cap V_j = \emptyset$ for every $j \in I \setminus L(z)$. By the continuity of f, it follows $f(z_n) \in U$ definitively and so $L(z_n) \subseteq L(z)$ definitively.

Now a set $L \in \beta$, containing $L(z)$, contains $L(z_n)$ as well, and the assertion follows by the definition of φ.

(e) To understand better the difficulties in the search of a fixed point of φ consider the extreme case when $V \in \mathcal{T}(0)$ is such that $\bigcap_{i \in I} V_i \neq \emptyset$. In this case $\beta = \{I\}$ and the values of φ are star-shaped.

Consequently φ has a fixed point by the Eilenberg-Montgomery theorem [8]. For a generic $V \in \mathcal{T}(0)$ and for a generic $x \in P$ the homology group $H^1(\varphi(x))$ is not trivial.

Unfortunately the usual fixed point theorems for multivalued maps, to our best knowledge, contain a drastic restriction on $H^1(\varphi(x))$ (see e.g. [2], [3], [4], [5], [6], [7], [10], [17], [18], [19] and the references listed therein).

So we are not able to apply any one of them to the map φ.

At any rate, if we were able to find fixed points for φ, we would find, in a standard way, a fixed point for f. In fact given a neighbourhood U of 0, fix $V \in \mathcal{T}(0)$ in such a way that $V - V \subseteq U$ and consider the map φ relative to V.

Let $x_U \in P$ be a point such that

$$x_U \in \varphi(x_U) \subseteq \cup\{V_i, i \in L(x_U)\}\,.$$

By the definition of $L(x_U)$ we have

$$f(x_U) \in \cap\{V_i, i \in L(x_U)\}$$

and on the other hand, there exists $i \in L(x_U)$ such that

$$f(x_U) \in V_i = (x_i + V) \cap D\,.$$

Consequently $x_U - f(x_U) \in V - V \subseteq U$, i.e. x_U is an almost fixed point for f. Since U is arbitrary, D is compact and f is continuous, f has a fixed point.

3. The set D is said to be locally convex if every point of D has a base of convex neighbourhoods in the relative topology.

Under the additional assumption that E is metrizable, Krauthausen proved that D (compact, convex and locally convex) has the fixed point property [15, Theorem 1.14] or [12, Theorem 3 on page 27]. And Weber [22] proved that the last statement holds without assuming the metrizability of E.

This result follows also using the techniques of the paragraph 2 (cf. also [13]). In fact, given $V \in \mathcal{T}(0)$, there exist convex subsets $U_1, U_2, \dots, U_n$ of V and $x_1, \dots, x_n \in D$ such that

$$D \subseteq \cup\{(x_i + U_i) \cap D, i \in I\}\,.$$

Using the same notations of paragraph 2, we define

$$\Psi(x) = \cup\{V_i \cap P, i \in L(x)\}\,,$$

where $V_i = (x_i + U_i) \cap D$.

Obviously Ψ enjoys the same continuity properties of φ, but, in contrast with φ, the values of Ψ are star-shaped (for $i \in L(x)$ the sets $V_i \cap P$ are convex and have in common all the points y_L with $L \in \beta$ and $L(x) \subseteq L$).

By the Eilenberg and Montgomery theorem [8], Ψ has a fixed point, which is an almost fixed point for f.

From the result, just exposed, it follows a result, now classic, due to Ky Fan ([9], [11]): if the space E has a separating dual (i.e. for every $x \in E \setminus \{0\}$ there exists a continuous linear functional such that $l(x) \neq 0$), then the Schauder Conjecture holds for E.

In fact in such a type of space, every compact convex subset D is locally convex. We like to point out that there exist also spaces with trivial dual, in which every compact convex subset is locally convex (for example, see Kalton [14]).

Finally we recall that, recently, Weber [22], in an ingenious way, has exhibited the first example, to our best knowledge, of a set D compact convex not locally convex and with the f.p.p.

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