On total curvature of immersions and minimal submanifolds of spheres

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Abstract. For closed immersed submanifolds of Euclidean spaces, we prove that $\int |\mu|^2\, dV \geq V/R^2$, where μ is the mean curvature field, V the volume of the given submanifold and R is the radius of the smallest sphere enclosing the submanifold. Moreover, we prove that the equality holds only for minimal submanifolds of this sphere.

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1. Introduction.

Let $x : M \to \mathbb{R}^3$ be an immersion of a smooth closed surface into the Euclidean space, with mean curvature function H. The total mean curvature of x is, by definition, the integral $\int H^2\, dV$ over M, where dV is the induced volume element. The idea of studying this integral, as a measure of the "niceness" of the shape of the immersed surface, was discussed at meetings in Oberwolfach in 1960 [9]. The first result on this subject was obtained by Willmore [8], which suggested the difficult problem of determining the infimum of the integral over all immersions, for a given M, and characterizing those immersions for which this minimum value is attained. Since then, the total mean curvature has become the object of intensive studies, giving rise to a vast research area, with many interesting open problems ([10], [2]).

Among the various possible generalizations of the concept of total mean curvature in higher dimensions and codimensions [4], one can consider the integral of the squared norm $|\mu|^2$ of the mean curvature vector field μ, for a given immersion $x : M^n \to \mathbb{R}^{n+p}$. In this paper we prove an extrinsic inequality relating this integral with a number which is sensitive to the shape of the immersed submanifold. More explicitly, we prove

$$\int_M |\mu|^2\, dV \geq V/R^2,$$

where V is the volume of the immersed submanifold and R denotes the radius of the smallest closed ball enclosing $x(M)$. Moreover, the equality holds if and only if x immerses M as a minimal submanifold into the Euclidean hypersphere bounding this ball.

The precise statement and proof of this result will be given in Section 3, after the preliminaries in Section 2; in Section 4 we will treat the case the hypersurfaces, and in Section 5 the case of the curves.

2. Notations and preliminary results.

Through this note, M denotes smooth (C^∞), connected, compact, oriented n-dimensional manifold without boundary. In Sections 2, 3 and 4 the dimension will be ≥ 2. Given a smooth immersion $x : M^n \to \mathbb{R}^{n+p}$ into Euclidean $(n+p)$-space, we assume that M is endowed with the Riemannian metric induced by x from the standard inner product $\langle , \rangle$ on $\mathbb{R}^{n+p}$. The volume n-form, volume and Laplace-Beltrami operator on M will be denoted by dV, V and Δ.

Let us recall the definition of the mean curvature normal field. Let ∇° be the Euclidean connection, and let ∇ be the induced Riemannian connection on M. If X, Y are vector fields on M, the following well-known *Gauss' formula* holds:

$$\nabla^\circ_X Y = \nabla_X Y + h(X, Y).$$

(Here, a vector field on M is automatically identified with its image by the differential x_*.) The normal component $h(X, Y)$ of the ambient covariant derivative is symmetric and bilinear in X, Y over the ring of $\mathbb{R}$-valued functions on M. The symmetric bilinear normal-bundle-valued function h is called the *second fundamental form* of the submanifold M, or of the immersion x. The normal vector field along x

$$\mu = (1/n) \text{ trace } (h)$$

is called the *mean curvature normal* of the immersed submanifold.

The following facts are all well-known. We state them for future use.

(i) $\Delta x = n\mu$. (See [6].)

(ii) Takahashi's theorem. *If x is a minimal immersion of M into the Euclidean $(n+p-1)$-sphere $S^{n+p-1}(O, R)$, with center at the origin O and radius R, then $\Delta x = -(n/r^2)x$. Conversely, if $\Delta x = \lambda x$, then λ is a negative constant and x is a minimal immersion of M into $S^{n+p-1}(O, R)$, where $R = \sqrt{(-n/\lambda)}$.* (See [2]. Recall that, if $x(M)$ lies in a sphere, then x is minimal into the sphere iff μ is purely normal.)

(iii) Minkowski's formula. $V = -\int_M \langle x, \mu \rangle \, dV$. (See [5].)

Let B be the smallest closed $(n+p)$-ball containing $x(M)$. By adapting the terminology of [1] to the present situation, we will call the radius and the center of B the *circumradius* and the *circumcenter* of $x(M)$, or of x. Without loss of generality, we can suppose that the circumcenter is O. Then the circumradius will be the maximum value of $|x|$ on M.

3. The main theorem.

We want to prove the following

Proposition 1. *If $x : M^n \to \mathbb{R}^{n+p}$ is a smooth immersion of a closed n-manifold, $n \geq 2$, then*

$$\int_M |\mu|^2 \, dV \geq V/R^2, \tag{1}$$

where R is the circumradius of $x(M)$. Moreover, the equality holds if and only if x is a minimal immersion of M into $S^{n+p-1}(O,R)$.

Proof: In the real vector space of $\mathbb{R}^n$-valued smooth functions on M, define the inner product of u and v by

$$(u,v) = \int_M \langle u, v\rangle \, dV.$$

Then the formula of Minkowski becomes $V = -(\mu, x)$, and Cauchy-Schwartz inequality gives $V^2 \leq (\mu,\mu)(x,x) \leq VR^2(\mu,\mu)$, which implies the inequality (1).

Now, if x minimally immerses M into $S^{n+p-1}(O,R)$, then the tangential component of μ must vanish, and μ coincides with $\pm(1/R^2)x$, i.e. the mean curvature normal of the standard $(n+p-1)$-sphere in $\mathbb{R}^{n+p}$. Consequently, the equality holds in (1).

Conversely, if the equality holds in (1), then $(\mu,x)^2 = (\mu,\mu)(x,x)$ and, by standard arguments, there exists $a \in \mathbb{R}$ such that $\mu = ax$. Therefore $\Delta x = n\mu = nax$, and the desired result follows by Takahashi's theorem. □

Remark 1. A famous result of Chern and Hsiung [3] says that there exist no compact minimal submanifolds in $\mathbb{R}^n$. This fact is also an immediate consequence of (1).

4. The case of hypersurfaces: a characterization of Euclidean hyperspheres.

Let us consider what happens when M^n, $n \geq 2$, is an immersed hypersurface, i.e. $p = 1$. Due to topological reasons, $x(M)$ can be contained in $S^n(O,R)$ only if it actually coincides with the whole sphere. In this case, Hadamard's theorem on ovaloids [6] forces x to be an imbedding. On the other hand the length of the mean curvature normal, up the sign, is the mean curvature function. Then we have the following characterization of Euclidean hyperspheres:

Proposition 2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an immersed closed hypersurface, with mean curvature function H, volume V, circumradius R and circumcenter O. Then*

$$\int_M H^2 \, dV \geq V/R^2 \tag{2}$$

and the equality holds if and only if x is an embedding and $x(M)$ coincides with the standard hypersphere $S^n(O,R)$.

Remark 2. If M^n is diffeomorphic to S^n (endowed with the standard differentiable structure) then, for given V/R^2, the immersion which realizes the minimum value of the integral (2) is the standard one. This circumstance agrees with the heuristic hypothesis of Willmore [8] on the aesthetic meaning of the total mean curvature.

5. The case of the curves.

In dealing with closed immersed 1-manifolds it is preferable to consider parametrized closed curves, rather than immersions of the circle S^1. Moreover, although the previous treatment can be adapted, with some changes, to the present situation, it seems more convenient to proceed directly.

A map $x : [0, L] \to \mathbb{R}^n$, $n \geq 2$ will be said a *nondegenerate closed curve of length* L if there exists a smooth map $y : \mathbb{R} \to \mathbb{R}^n$ such that

(i) y has period L and $y \mid [0, L] = x$,
(ii) $|y'(s)| = 1$ for all $s \in \mathbb{R}$, and
(iii) there exists a Frenet-n-frame along y.

The *curvature* of x is the restriction k at $[0, L]$ of the (first) curvature of y.

Proposition 3. *Let $x : [0, L] \to \mathbb{R}^n$, $n \geq 2$, be a nondegenerate closed curve of length L, with curvature k, circumradius R and circumcenter O. Then we have*

$$\int_0^L k^2\, ds \geq L/R^2. \tag{3}$$

Moreover, the equality holds if and only if $x([0, L])$ is the circle $S^1(0, L)$, covered once by x.

Proof: In our hypotheses we have

$$L = \int_0^L |x'|^2\, ds = -\int_0^L \langle x, x'' \rangle\, ds.$$

Then, applying Cauchy-Schwartz inequality for integrals, we obtain

$$L^2 \leq \int_0^L |x|^2\, ds \int_0^L |x''|^2\, ds,$$

which implies the inequality (3). Now, if x maps $[0, L]$ onto $S^1(O, R)$, without double points in $]0, L[$, we have, of course, the equality in (3). Conversely, if the equality holds, then must be $x'' = ax$, $a \in \mathbb{R}$. Thus, following Chen [2], x is a closed curve of 1-type and, consequently, $x([0, L])$ lies in a plane; but then it is a circle, and the result follows easily. □

Remark 3. In [7] Weiner proved an inequality analogous to (3), although much more involved. For plane curves, the two inequalities coincide.

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