Commutative neutrix convolution products of functions

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Abstract. The commutative neutrix convolution product of the functions $x^r e_-^{\lambda x}$ and $x^s e_+^{\mu x}$ is evaluated for $r, s = 0, 1, 2, \dots$ and all λ, μ. Further commutative neutrix convolution products are then deduced.

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In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f*g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined by the equation

$$\langle (f*g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side,

see Gel'fand and Shilov [7].

Note that if f and g are locally summable functions satisfying either of the above conditions then

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt. \tag{1}$$

It follows that if the convolution product $f*g$ exists by this definition then

$$f*g = g*f, \tag{2}$$

$$(f*g)' = f*g' = f'*g. \tag{3}$$

This definition of the convolution product is rather restrictive and so the non-commutative neutrix convolution product was introduced in [2]. A commutative neutrix convolution product was given more recently in [4]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \dots$.

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ and $g_n = g\tau_n$ for $n = 1, 2, \dots$. Then the commutative neutrix convolution product $f \boxast g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided that the limit h exists in the sense that*

$$\operatorname*{N-lim}_{n\to\infty} \langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], *having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n, \ \ln^r n \quad (\lambda > 0, \ r = 1, 2, \dots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that in this definition the convolution product $f_n * g_n$ is defined in Gel'fand and Shilov's sense, the distributions f_n and g_n both having bounded support. Note also that the non-commutative neutrix convolution, denoted by $f \circledast g$, was defined as the limit of the sequence $\{f_n * g\}$.

The following theorem was proved in [4], showing that the neutrix convolution product is a generalization of the convolution product.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition* (a) *or condition* (b) *of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \boxast g$ exists and*

$$f \boxast g = f * g.$$

A number of neutrix convolution products have been evaluated. For example, $x_-^\lambda \boxast x_+^\mu$ see [4], $x_-^\lambda \boxast x_+^{r-\lambda}$ see [5] and $\ln x_- \boxast x_+^r$ see [6].

In order to define further neutrix convolution products, we increase our set of negligible functions given in Definition 1 to also include finite linear sums of the functions

$$n^\lambda e^{\mu n} \quad (\mu > 0).$$

We now define the locally summable functions $e_+^{\lambda x}$ and $e_-^{\lambda x}$ by

$$e_+^{\lambda x} = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad e_-^{\lambda x} = \begin{cases} 0, & x > 0, \\ e^{\lambda x}, & x < 0. \end{cases}$$

It follows that

$$e_-^{\lambda x} + e_+^{\lambda x} = e^{\lambda x}, \quad x^r e_+^{\lambda x} = x_+^r e_+^{\lambda x}, \quad x^r e_-^{\lambda x} = (-1)^r x_-^r e_-^{\lambda x},$$

for $r = 0, 1, 2, \dots$.

We now prove

Theorem 2. *The neutrix convolution product* $(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\mu x})$ *exists and*

$$e_-^{\lambda x} \boxast e_+^{\mu x} = \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu}, \tag{4}$$

$$\begin{aligned}(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\mu x}) &= D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu} \\ &= \sum_{i=0}^{s} \binom{s}{i} \frac{(r+s-i)!x^i e_+^{\mu x}}{(\lambda-\mu)^{r+s-i+1}} + \\ &\quad + \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i}(r+s-i)!x^i e_-^{\lambda x}}{(\lambda-\mu)^{r+s-i+1}},\end{aligned} \tag{5}$$

where $D_\lambda = \partial/\partial\lambda$ *and* $D_\mu = \partial/\partial\mu$, *for* $\lambda \neq \mu$ *and* $r, s = 0, 1, 2, \dots$; *these neutrix convolution products existing as convolution products if* $\lambda > \mu$ *and*

$$(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\lambda x}) = -B(r+1, s+1)\operatorname{sgn} x . x^{r+s+1} e^{\lambda x}, \tag{6}$$

where B *denotes the Beta function, for all* λ *and* $r, s = 0, 1, 2, \dots$.

Proof: We put $(e_-^{\lambda x})_n = e_-^{\lambda x}\tau_n(x)$ for $n = 1, 2, \dots$ and suppose first of all that $\lambda \neq \mu$. Since $(e_-^{\lambda x})_n$ and $(e_+^{\mu x})_n$ are summable functions with compact support, the convolution product $(e_-^{\lambda x})_n * (e_+^{\mu x})_n$ is defined by equation (1) and so

$$(e_-^{\lambda x})_n * (e_+^{\mu x})_n = \int_{-\infty}^{\infty} (e_-^{\lambda t})_n (e_+^{\mu(x-t)})_n \, dt = \int_{-n-n^{-n}}^{0} e^{\lambda t}\tau_n(t) e_+^{\mu(x-t)} \tau_n(x-t)\, dt.$$

Thus if $-n \le x \le 0$,

$$\begin{aligned}(e_-^{\lambda x})_n * (e_+^{\mu x})_n &= \int_{-n}^{x} e^{\lambda t} e^{\mu(x-t)}\, dt + \int_{-n-n^{-n}}^{-n} e^{\lambda t}\tau_n(t) e^{\mu(x-t)}\tau_n(x-t)\, dt \\ &= \frac{e^{\lambda x} - e^{\mu x - (\lambda-\mu)n}}{\lambda - \mu} + O(n^{-n} e^{-(\lambda-\mu)n}).\end{aligned} \tag{7}$$

When $n \ge x \ge 0$,

$$\begin{aligned}(e_-^{\lambda x})_n * (e_+^{\mu x})_n &= \int_{x-n}^{0} e^{\lambda t} e^{\mu(x-t)}\, dt + \int_{x-n-n^{-n}}^{x-n} e^{\lambda t}\tau_n(t) e^{\mu(x-t)}\tau_n(x-t)\, dt \\ &= \frac{e^{\mu x} - e^{\lambda x - (\lambda-\mu)n}}{\lambda - \mu} + O(n^{-n} e^{-(\lambda-\mu)n}).\end{aligned} \tag{8}$$

It now follows from equations (7) and (8) that for arbitrary ϕ in $\mathcal{D}$

$$\begin{aligned}\langle (e_-^{\lambda x})_n * (e_+^{\mu x})_n, \phi(x)\rangle &= (\lambda-\mu)^{-1}\langle e_+^{\mu x} + e_-^{\lambda x}, \phi(x)\rangle + \\ &\quad - (\lambda-\mu)^{-1} e^{-(\lambda-\mu)n}\langle e_+^{\lambda x} + e_-^{\mu x}, \phi(x)\rangle + O(n^{-n} e^{-(\lambda-\mu)n})\end{aligned}$$

and so

$$\operatorname*{N-lim}_{n\to\infty}\langle (e_-^{\lambda x})_n * (e_+^{\mu x})_n, \phi(x)\rangle = (\lambda-\mu)^{-1}\langle e_+^{\mu x} + e_-^{\lambda x}, \phi(x)\rangle,$$

the usual limit existing if $\lambda > \mu$. Equation (4) follows.

We now put $(x^r e_-^{\lambda x})_n = x^r e_-^{\lambda x}\tau_n(x)$ and $(x^s e_+^{\mu x})_n = x^s e_+^{\mu x}\tau_n(x)$. Then as above, we have

$$(x^r e_-^{\lambda x})_n * (x^s e_+^{\mu x})_n = \int_{-n-n^{-n}}^{0} t^r e^{\lambda t}\tau_n(t)(x-t)^s e_+^{\mu(x-t)}\tau_n(x-t)\,dt.$$

Thus if $-n \le x \le 0$,

$$\begin{aligned}(x^r e_-^{\lambda x})_n * (x^s e_+^{\mu x})_n &= \int_{-n}^{x} t^r e^{\lambda t}(x-t)^s e^{\mu(x-t)}\,dt + \\ &\quad + \int_{-n-n^{-n}}^{-n} t^r e^{\lambda t}\tau_n(t)(x-t)^s e^{\mu(x-t)}\tau_n(x-t)\,dt \\ &= D_\lambda^r D_\mu^s\, e^{\mu x}\int_{-n}^{x} e^{(\lambda-\mu)t}\,dt + O(n^{-n+r+s}e^{-(\lambda-\mu)n}) \\ &= D_\lambda^r D_\mu^s\,\frac{e^{\lambda x}}{\lambda-\mu} + e^{\mu x}P(n)\cdot e^{-(\lambda-\mu)n} + \\ &\qquad + O(n^{-n+r+s}e^{-(\lambda-\mu)n}),\end{aligned} \tag{9}$$

on using equation (7), where P denotes a polynomial.

When $n \ge x \ge 0$,

$$\begin{aligned}(x^r e_-^{\lambda x})_n * (x^s e_+^{\mu x})_n &= \int_{x-n}^{0} t^r e^{\lambda t}(x-t)^s e^{\mu(x-t)}\,dt + \\ &\quad + \int_{x-n-n^{-n}}^{x-n} t^r e^{\lambda t}\tau_n(t)(x-t)^s e^{\mu(x-t)}\tau_n(x-t)\,dt \\ &= D_\lambda^r D_\mu^s e^{\mu x}\int_{x-n}^{0} e^{(\lambda-\mu)t}\,dt + O(n^{-n+r+s}e^{-(\lambda-\mu)n}) \\ &= D_\lambda^r D_\mu^s\,\frac{e^{\mu x}}{\lambda-\mu} + e^{\lambda x}P(n)e^{-(\lambda-\mu)n} + \\ &\qquad + O(n^{-n+r+s}e^{-(\lambda-\mu)n}),\end{aligned} \tag{10}$$

on using equation (8).

It now follows as above from equations (9) and (10) that for arbitrary ϕ in $\mathcal{D}$

$$\operatorname*{N-lim}_{n\to\infty}\langle (x^r e_-^{\lambda x})_n * (x^s e_+^{\mu x})_n, \phi(x)\rangle = D_\lambda^r D_\mu^s(\lambda-\mu)^{-1}\langle e_+^{\mu x} + e_-^{\lambda x}, \phi(x)\rangle,$$

the usual limit existing if $\lambda > \mu$. Thus

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu}$$

and equation (5) follows.

Now suppose that $\lambda = \mu$. Then as above, we have

$$(x^r e_-^{\lambda x})_n * (x^s e_+^{\lambda x})_n = \int_{-n-n^{-n}}^{0} t^r e^{\lambda t} \tau_n(t)(x-t)^s e_+^{\lambda(x-t)} \tau_n(x-t)\, dt.$$

Thus if $-n \le x \le 0$,

$$\begin{aligned}
&(x^r e_-^{\lambda x})_n * (x^s e_+^{\lambda x})_n = \\
&= e^{\lambda x} \int_{-n}^{x} t^r (x-t)^s \, dt + e^{\lambda x} \int_{-n-n^{-n}}^{-n} t^r \tau_n(t)(x-t)^s \tau_n(x-t)\, dt \\
&= e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} (-1)^i \int_{-n}^{x} x^{s-i} t^{r+i}\, dt + O(n^{-n+r+s}) \\
&= e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} (-1)^i \frac{x^{r+s+1} - (-n)^{r+i+1} x^{s-i}}{r+i+1} + O(n^{-n+r+s}) \qquad (11)\\
&= e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} x^{r+s+1} (-1)^i \int_0^1 t^{r+i}\, dt + e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^r x^{s-i} n^{r+i+1}}{r+i+1} + \\
&\qquad + O(n^{-n+r+s}) \\
&= B(r+1, s+1) x^{r+s+1} e^{\lambda x} + e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^r x^{s-i} n^{r+i+1}}{r+i+1} + \\
&\qquad + O(n^{-n+r+s}),
\end{aligned}$$

where B denotes the Beta function.

When $x \ge 0$,

$$\begin{aligned}
&(x^r e_-^{\lambda x})_n * (x^s e_+^{\lambda x})_n = \\
&\qquad = e^{\lambda x} \int_{x-n}^{0} t^r (x-t)^s \, dt + e^{\lambda x} \int_{x-n-n^{-n}}^{x-n} t^r (x-t)^s \tau_n(t)\, dt \\
&\qquad = e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^{i+1} x^{s-i} (x-n)^{r+i+1}}{r+i+1} + O(n^{-n+r+s})
\end{aligned}$$

and it follows that

$$\begin{aligned}
\underset{n\to\infty}{\text{N}-\lim} (x^r e_-^{\lambda x})_n * (x^s e_+^{\lambda x})_n &= x^{r+s+1} e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^{i+1}}{r+i+1} \qquad (12)\\
&= -B(r+1, s+1) x^{r+s+1} e^{\lambda x},
\end{aligned}$$

when $x \geq 0$.

It now follows as above from equations (11) and (12) that for arbitrary ϕ in $\mathcal{D}$

$$\operatorname*{N-lim}_{n\to\infty}\langle (x^r e_-^{\lambda x})_n * (x^s e_+^{\lambda x}), \phi(x)\rangle = B(r+1, s+1)\langle x^{r+s+1} e_-^{\lambda x} - x^{r+s+1} e_+^{\lambda x}, \phi(x)\rangle$$

and equation (6) follows.

Corollary. *The neutrix convolution products* $(x^r e^{\lambda x}) \boxed{*} (x^s e_\pm^{\mu x})$ *and* $(x^r e^{\lambda x}) \boxed{*} (x^s e^{\mu x})$ *exist and*

$$(x^r e^{\lambda x}) \boxed{*} (x^s e_\pm^{\mu x}) = \pm D_\lambda^r D_\mu^s \frac{e^{\lambda x}}{\lambda - \mu}, \tag{13}$$

$$(x^r e^{\lambda x}) \boxed{*} (x^s e^{\mu x}) = 0, \tag{14}$$

for $\lambda \neq \mu$ *and* $r, s = 0, 1, 2, \ldots$ *and*

$$(x^r e^{\lambda x}) \boxed{*} (x^s e_\pm^{\lambda x}) = \pm B(r+1, s+1) x^{r+s+1} e_\mp^{\lambda x}, \tag{15}$$

$$(x^r e^{\lambda x}) \boxed{*} (x^s e^{\lambda x}) = -B(r+1, s+1)\operatorname{sgn} x . x^{r+s+1} e^{\lambda x}, \tag{16}$$

for all λ *and* $r, s = 0, 1, 2, \ldots$.

Proof: We will suppose first of all that $\lambda \neq \mu$. It was proved in [3] that

$$(x^r e_+^{\lambda x}) * (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\lambda x} - e_+^{\mu x}}{\lambda - \mu}, \tag{17}$$

$$(x^r e_-^{\lambda x}) * (x^s e_-^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_-^{\lambda x} - e_-^{\mu x}}{\mu - \lambda}. \tag{18}$$

It follows that

$$(x^r e^{\lambda x}) \boxed{*} (x^s e_+^{\mu x}) = (x^r e_+^{\lambda x} + x^r e_-^{\lambda x}) \boxed{*} (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e^{\lambda x}}{\lambda - \mu},$$

on using equations (5) and (17) and noting that the neutrix convolution product is distributive with respect to addition.

Similarly,

$$(x^r e^{\lambda x}) \boxed{*} (x^s e_-^{\mu x}) = (x^r e_+^{\lambda x} + x^r e_-^{\lambda x}) \boxed{*} (x^s e_-^{\mu x}) = -D_\lambda^r D_\mu^s \frac{e^{\lambda x}}{\lambda - \mu},$$

on using equations (5) and (18). Equations (13) are proved.

We now have

$$(x^r e^{\lambda x}) \boxed{*} (x^s e^{\mu x}) = (x^r e^{\lambda x}) \boxed{*} (x^s e_+^{\mu x} + x^s e_-^{\mu x}) = 0,$$

on using equations (13), proving equation (14).

Now suppose that $\lambda = \mu$. It was proved in [3] that in this case

$$(x^r e_+^{\lambda x}) * (x^s e_+^{\lambda x}) = B(r+1, s+1) x^{r+s+1} e_+^{\lambda x}, \tag{19}$$

$$(x^r e_-^{\lambda x}) * (x^s e_-^{\lambda x}) = -B(r+1, s+1) x^{r+s+1} e_-^{\lambda x}. \tag{20}$$

It follows that

$$(x^r e^{\lambda x}) \boxast (x^s e_+^{\lambda x}) = (x^r e_+^{\lambda x} + x^r e_-^{\lambda x}) \boxast (x^s e_+^{\lambda x}) = B(r+1, s+1) x^{r+s+1} e_-^{\lambda x} \tag{21}$$

on using equations (5) and (19).

Similarly,

$$(x^r e^{\lambda x}) \boxast (x^s e_-^{\lambda x}) = (x^r e_+^{\lambda x} + x^r e_-^{\lambda x}) * (x^s e_-^{\lambda x}) = -B(r+1, s+1) x^{r+s+1} e_+^{\lambda x}, \tag{22}$$

on using equations (5) and (20) and then

$$(x^r e^{\lambda x}) \boxast (x^s e^{\lambda x}) = (x^r e^{\lambda x}) \boxast (x^s e_+^{\lambda x} + x^s e_-^{\lambda x}) = -B(r+1, s+1) \operatorname{sgn} x . x^{r+s+1} e^{\lambda x},$$

on using equations (21) and (22). Equations (15) and (16) are now proved.

The non-commutative neutrix convolution product $(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x})$ was evaluated in [3]. Note that

$$(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\mu x}) = (x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x}),$$

for $\lambda \neq \mu$, but

$$(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\lambda x}) \neq (x^r e_-^{\lambda x}) \circledast (x^s e_+^{\lambda x}).$$

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