Comment.Math.Univ.Carolin. 35,1 (1994)169–179 

# Expected values in the alternative set theory and their applications to some limit theorems

Martin Kalina

*Abstract.* This article presents an alternative approach to statistical moments within non-standard models and by the help of these moments some limit theorems are reformulated and proved.

*Keywords:* alternative set theory, expected value, law of large numbers, central limit theorem

*Classification:* 26E35, 60A99, 60F05

The aim of this paper is to introduce a new approach to statistical moments and to show, on some limit theorems, how they are to be used. It is written in the framework of the Alternative Set Theory. Basic notions and notation from [V] will be used freely, without any reference. The author has been inspired to write this article by Nelson's book [N].

If we have a random variable $\xi$, then its mean is usually defined by

$$m(\xi) = \sum_{\omega\in\Omega} \xi(\omega) \cdot p(\omega),$$

where $\Omega$ is the domain of $\xi$ and $p$ is its probability distribution. But even certain fluctuations of an infinitesimal probability can cause a considerable change of this mean. As well-known, the so called normal probability distribution depends on two parameters, $E$ and $\sigma^2$. Let us call them the expected value and the dispersion, respectively. From the Central Limit Theorem it follows that if we have a sequence of independent random variables $\{\eta_i\}_{i=1}^{\beta}$ (with some particular properties), $\beta$ being infinite, then the probability distribution of the sum of $\eta_i$, suitably normed (i.e. the first and the second "statistical moment" of the sum are 0 and 1, respectively), is "very close" to the normal probability distribution with its expected value 0 and dispersion 1. But, as we shall see below, though the first "statistical moment" of the normed sum is equal to 0, we have no idea about the value of its mean, defined as above. So, there is really the need for introducing of statistical moments, different from those, already defined.

Preliminarily, let us give some notions and notations, valid in the whole paper. First, under a random variable we will understand a set function with a nonempty domain, denoted by $\Omega$, and the set $\Omega$ will be equipped with the uniform probability distribution, i.e. $p(\{\omega\}) = 1/\alpha$ for each $\omega \in \Omega$, where $\alpha$ is the number of the elements of $\Omega$. The letters $\xi$ and $\eta$, possibly indexed, will always denote random variables. The common range of all random variables, used in this paper, will be denoted by $u$.

In order to be short, $p(\{\omega \in \Omega; a \leq \xi(\omega) \leq b\})$ will be denoted by $p(a \leq \xi \leq b)$, or simply by $p([a,b])$ (for any $a < b \in \mathbb{Q}$), $p(\{\omega \in \Omega; \xi(\omega) = x\})$ will be denoted by $p(\xi = x)$, or simply by $p(x)$ (for any $x \in u$), etc., wherever it will not cause any confusions.

A random variable $\xi$ will be called **standard**, if for each $\gamma \in \mathbb{N} \setminus \mathbb{FN}$ $p(-\gamma \leq \xi \leq \gamma) \doteq 1$ holds.

For any $x \in \mathbb{Q}$ the class $\{y \in \mathbb{Q}; x \doteq y\}$ will be denoted by $\mathrm{mon}(x)$. The probability $p$ can be extended in the sense of Loeb (for more details see [L] or [K-Z]). This extension will be denoted by $P$. So, particularly, the expression $P(\xi \in \mathrm{mon}(x))$ is a real number for each random variable $\xi$ and each $x \in \mathbb{Q}$.

Finally, we will denote

$$\begin{array}{llllll} a \simeq b & \text{if} & a/b \doteq 1 & & \\ a \sim b & \text{if} & a/b \in \mathbb{BQ} & \text{and} & b/a \in \mathbb{BQ} \\ a \lesssim b & \text{if} & a < b & \text{or} & a \sim b \\ a \lnsim b & \text{if} & a < b & \text{and} & a \nsim b \end{array}$$

for any nonzero numbers $a, b \in \mathbb{Q}$.

Now, we are ready to investigate the announced statistical moments. We have already defined the mean of a random variable $\xi$. More generally, for any random variable $\xi$ and $k \in \mathbb{FN}$ we denote

$$m_k(\xi) = \frac{1}{\alpha} \sum_{\omega \in \Omega} \xi^k(\omega)$$

(let us remind that $\Omega = dom(\xi)$ and $\alpha$ is the number of its elements).

**Lemma 1.** *Let $0 \neq k \in \mathbb{FN}$ and $m_{2k}(\xi) \in \mathbb{BQ}$. Then $\xi$ is a standard random variable.*

Proof: Let $\xi$ not be standard. Then there exist $\beta, \gamma$, $\beta > \gamma \in \mathbb{N} \setminus \mathbb{FN}$, such that

$$p([-\beta, -\gamma]) + p([\gamma, \beta]) \not\doteq 0,$$

hence

$$m_{2k}(\xi) \geq \sum_{\substack{x \in u \\ -\beta \leq x \leq -\gamma}} x^{2k} \cdot p(x) + \sum_{\substack{x \in u \\ \gamma \leq x \leq \beta}} x^{2k} \cdot p(x) \geq \gamma^{2k} \cdot (p([-\beta, -\gamma]) + p([\gamma, \beta])),$$

and we get $m_{2k}(\xi) \notin \mathbb{BQ}$, and that is a contradiction. $\square$

Random variables $\xi$ and $\eta$ are said to be **infinitesimally shifted** if they yield the following properties for all $x \in \mathbb{BQ}$

(1) $\quad P(\xi \in \mathrm{mon}(x)) = P(\eta \in \mathrm{mon}(x))$

(2) $\quad p(\xi \leq x) \doteq p(\eta \leq x), \qquad$ if $\quad P(\xi \in \mathrm{mon}(x)) = 0$

Let $\xi$ be a standard random variable, $0 \neq k \in \mathbb{FN}$. If there exists a $\gamma \in \mathbb{N} \setminus \mathbb{FN}$ such that each set $v \subset \mathbb{Q}$ yields

$$p(v) \doteq 0 \Rightarrow \sum_{\substack{x \in u \\ -\gamma \leq x \leq \gamma}} x^k p(x) - \sum_{\substack{x \in u \setminus v \\ -\gamma \leq x \leq \gamma}} x^k p(x) \doteq 0,$$

then the real number

$$E_k(\xi) = \mathrm{mon}( \sum_{\substack{x \in u \\ -\gamma \leq x \leq \gamma}} x^k p(x))$$

will be called the **expected value of the k-th order** of $\xi$.

In spite of the fact that $E_k(\xi)$ is a real number and $m_k(\xi) \in \mathbb{Q}$, we will write $E_k(\xi) = m_k(\xi)$, if $\mathrm{mon}(m_k(\xi)) = E_k(\xi)$. Similarly we will write $E_k(\xi) \leq m_k(\xi)$, etc.

It is just an easy exercise to show that $E_k(\xi)$ is always finite, if it exists.

**Theorem 1.** *Let $\xi$ and $\eta$ be standard random variables, having their probability distributions infinitesimally shifted. Suppose there exists the expected value of the k-th order of $\xi$. Then there exists the expected value of the k-th order of $\eta$ and $E_k(\xi) = E_k(\eta)$.*

PROOF: For any $n \in \mathbb{N}$ denote

$$\hat{\xi}_n = \sum_{\substack{x \in u \\ -n \leq x \leq n}} x^k \cdot p(\xi = x), \quad \hat{\eta}_n = \sum_{\substack{x \in u \\ -n \leq x \leq n}} x^k \cdot p(\eta = x)$$

(let us remind that $u$ is the common range of $\xi$ and $\eta$). Since the probability distributions of $\xi$ and $\eta$ are infinitesimally shifted, we have $\hat{\xi}_n \doteq \hat{\eta}_n$ for each $n \in \mathbb{FN}$, hence, by Prolongation Axiom, there is a $\gamma \in \mathbb{N} \setminus \mathbb{FN}$ such that each $\beta \leq \gamma$, $\beta$ being infinite, yields $\hat{\xi}_\beta \doteq \hat{\eta}_\beta$, and this implies the existence of $E_k(\eta)$ and the equality $E_k(\eta) = E_k(\xi)$. □

**Theorem 2.** *Let $\xi$ and $\eta$ be standard random variables and let there exist their expected values of all finite orders and, moreover, $E_k(\xi) = E_k(\eta)$ for all $k \in \mathbb{FN}$. Then their probability distributions are infinitesimally shifted.*

PROOF: From the definition of the expected value we get the existence of such a $\gamma \in \mathbb{Q}$ that for all $k \in \mathbb{FN}$ there holds

$$\sum_{\substack{x \in u \\ -\gamma \leq x \leq \gamma}} x^k \cdot p(\xi = x) \doteq \sum_{\substack{x \in u \\ -\gamma \leq x \leq \gamma}} x^k \cdot p(\eta = x).$$

Hence, for any $n \in \mathbb{FN}$ and any $a_k \in \mathbb{BQ}$, $0 \le k \le n$ we get

$$\sum_{k=0}^{n} a_k \sum_{\substack{x\in u \\ -\gamma\le x\le\gamma}} x^k \cdot p(\xi = x) \doteq \sum_{k=0}^{n} a_k \sum_{\substack{x\in u \\ -\gamma\le x\le\gamma}} x^k \cdot p(\eta = x).$$

Since the last formula holds for every polynomial $\sum_{k=0}^{n} a_k x^k$, the assertion in question is proved.

□

Hence the expected values have all basic properties of the statistical moments in the "classical" probability theory. Something about the relationship between the moments $E_k$ and $m_k$ we shall learn from the following

**Theorem 3.** *Let $\xi$ be any random variable, $0 \ne n \in \mathbb{FN}$ and $m_{2n}(\xi) \in \mathbb{BQ}$. Then $\xi$ is a standard random variable, whose expected values exist up to the $2n$-th order, $m_k(\xi) = E_k(\xi)$ for all $k \le 2n-1$ and $E_{2n}(\xi) \le m_{2n}(\xi)$.*

Proof: $\xi$ is a standard random variable due to Lemma 1. Let for a $k < 2n$ $m_k(\xi) \ne E_k(\xi)$. Then, obviously, there exist infinite numbers $\beta < \gamma$ such that

$$\sum_{\substack{x\in u \\ \beta\le x\le\gamma}} x^k \cdot p(x) \not\doteq 0 \quad \text{or} \quad \sum_{\substack{x\in u \\ -\gamma\le x\le -\beta}} x^k \cdot p(x) \not\doteq 0.$$

Then

$$m_{2n}(\xi) \ge \sum_{\substack{x\in u \\ \beta\le |x|\le\gamma}} x^{2n} \cdot p(x) \ge \beta^{2n-k} \cdot \sum_{\substack{x\in u \\ \beta\le |x|\le\gamma}} x^k \cdot p(x) \notin \mathbb{BQ},$$

and we get a contradiction, hence $m_k(\xi) = E_k(\xi)$ for all $k < n$.

Denote $f(i) = \sum_{\substack{x\in u \\ |x|\le i}} x^{2n} \cdot p(x)$. Since $f(i)$, for $i \in \mathbb{FN}$, is a nondecreasing, bounded above function, we get the existence of $E_{2n}(\xi)$. The inequality $E_{2n}(\xi) \le m_{2n}(\xi)$ is obvious. □

So, particularly, if $m_n(\xi)$ is bounded[1] for each $n \in \mathbb{FN}$, then $E_n(\xi) = m_n(\xi)$ holds for each $n \in \mathbb{FN}$.

Now, we have a system of independent random variables $\{\xi_i\}_{i=1}^n$,[2] and we are interested in how to compute the expected value of a given order of the sum of

[1]that means it is an element of $\mathbb{BQ}$

[2]that is $p(\xi_1 \in v_1 \;\; \& \;\; \xi_2 \in v_2 \;\; \& \;\; \dots \;\; \& \;\; \xi_n \in v_n) = p(\xi_1 \in v_1) \cdot p(\xi_2 \in v_2) \cdots p(\xi_n \in v_n)$ for any set $v_1 \times v_2 \times \dots \times v_n \subseteq \mathbb{Q}^n$.

our system. Of course, this task is trivial in case our system is finite, and we are not going to solve it. But what to do in case $n$ is infinite? Remind that, by our definition, by computing of the expected value we neglect some set of an infinitesimal probability, so if we had simply used the formulas known for the finite system, then the union of all neglected sets would possibly have been of a positive probability. So we have to develop another technique in order to manage our task.

Let $\delta \in \mathbb{N}$ and $a, b \in \mathbb{Q}$. We denote $a =_\delta b$, if $(a-b)\cdot\delta \doteq 0$. Particularly, if $a \doteq b$, then $a =_1 b$. The class $\{\gamma \in \mathbb{Q}; a =_\delta \gamma\}$ will be denoted by $\mathrm{mon}_\delta(a)$ and called the **$\delta$-number**.

We will write $a \simeq b$, $a \sim b$, etc, also for $\delta$-numbers in the sense that the formula holds for all their representatives.

In the remainder of this paper, the letters $\beta$, $\gamma$, $\delta$ will always denote infinite natural numbers.

If $\{\gamma_i\}_{i=1}^{\beta}$ is a sequence of $\delta$-numbers, $\beta \lesssim \delta$, then we can define the sum of our system in the usual sense, i.e. as the sum of some representatives of our $\delta$-numbers. Of course, this sum will not be a $\delta$-number, but a $\delta/\beta$-number. Particularly, in case $\beta \sim \delta$ and the sum is bounded, it will be a real number.

Let $\xi$ be any random variable and $\delta \in \mathbb{N}$. If for a $\gamma \in \mathbb{N}$, such that $p(\xi \in [-\gamma,\gamma]) =_\delta 1$, and for each set $v \subset \mathbb{Q}$ with $p(\xi \in v) =_\delta 0$

$$\sum_{\substack{x\in u\\ -\gamma\le x\le\gamma}} x^k p(x) =_\delta \sum_{\substack{x\in u\setminus v\\ -\gamma\le x\le\gamma}} x^k p(x),$$

then the $\delta$-number $\mathrm{mon}_\delta(\sum_{\substack{x\in u\\ -\gamma\le x\le\gamma}} x^k p(x))$ will be called the **$\delta$-expected value of the k-th order** of $\xi$ and denoted by $\tilde{\delta}_k(\xi)$.

For $\gamma \le \delta \in \mathbb{N}$ we will write $\tilde{\delta}_k(\xi) = \tilde{\gamma}_k(\xi)$ instead of $\tilde{\delta}_k(\xi) \subseteq \tilde{\gamma}_k(\xi)$. Similarly, we will write $\tilde{\delta}_k(\xi) \le \tilde{\gamma}_k(\xi)$, etc.

The relationship between $\tilde{\delta}_k(\xi)$ and $\tilde{\gamma}_k(\xi)$ is the following

**Lemma 2.** *Let $\xi$ be any random variable and $\gamma \lesssim \delta$. Then*

- **(a)** *if $0 \ne n \in \mathbb{FN}$ and $\tilde{\delta}_{2n}(\xi)$ is bounded, then $\xi$ is a standard random variable, whose $\gamma$-expected values exist up to the 2n-th order, $\tilde{\delta}_k(\xi) = E_k(\xi)$ for all $k \le 2n-1$ and $\tilde{\gamma}_{2n}(\xi) \le \tilde{\delta}_{2n}(\xi)$.*
- **(b)** *If there exists a set $v$ with $p(\xi \in v) \lnsim 1/\delta$ such that $\sum_{x\in u\setminus v} \xi^n(x)\cdot p(x)$ is bounded for each $n \in \mathbb{N}$, then*

  $$\sum_{x\in u\setminus v} \xi^n(x)\cdot p(x) = \tilde{\gamma}_n(\xi)$$

  *holds for each $n \in \mathbb{N}$.*

*Proof* of these assertions differs just slightly from that of Theorem 3, therefore it is omitted.

**Theorem 4.** *Let $\{\xi_i\}_{i=0}^{\delta}$ be a sequence of independent random variables with $\tilde{\delta}_1(\xi_i) = 0$ (0 is meant as a $\delta$-number!) and $\tilde{\delta}_2(\xi_i) \lesssim 1/\delta$ for each $i \leq \delta$, then the expected value of the first order of the sum of $\{\xi_i\}_{i=0}^{\delta}$ is equal to 0 and* $E_2(\sum_{i=0}^{\delta} \xi_i) = \sum_{i=0}^{\delta} \tilde{\delta}_2(\xi_i)$.

Proof: Let us compute $E_2(\sum_{i=0}^{\delta} \xi_i)$, provided it exists. Obviously,

$$E_2(\sum_{i=0}^{\delta} \xi_i) \leq \sum_{i=0}^{\delta} \tilde{\delta}_2(\xi_i) + 2 \cdot \sum_{i=1}^{\delta} \sum_{j=0}^{i-1} \tilde{\delta}_1(\xi_i) \cdot \tilde{\delta}_1(\xi_j), \tag{3}$$

since on the right-hand side we have neglected just a set of measure zero. By our assumptions, $\sum_{i=0}^{\delta} \tilde{\delta}_2(\xi_i)$ is bounded and $\sum_{i=1}^{\delta} \sum_{j=0}^{i-1} \tilde{\delta}_1(\xi_i) \cdot \tilde{\delta}_1(\xi_j) = 0$, hence, by Lemma 2a, the expected value of the second order of our sum does exist and, hence, so does the expected value of the first order. Of course, on the left-hand side of (3) we can neglect any set of measure zero. That means, we can choose a subsequence $\{\xi_{i_j}\}_{j=0}^{\gamma}$, $\gamma \lnsim \delta$ and by each of our chosen random variables neglect a set $v_{i_j}$ with $p(\xi_{i_j} \in v_{i_j}) \gtrsim 1/\delta$ and $\sum_{j=0}^{\gamma} p(\xi_{i_j} \in v_{i_j}) \doteq 0$. But, since $\sum_{j=0}^{\gamma} \tilde{\delta}_2(\xi_{i_j}) = 0$, we get the equality in (3).

Further we have $0 = \sum_{i=0}^{\delta} \tilde{\delta}_1(\xi_i)$, and $E_1\left(\sum_{i=0}^{\delta}(\xi_i)\right) = \sum_{i=0}^{\delta} \sum_{x \in v_i} x \cdot p(\xi_i = x)$, where for a chosen subsequence $\{v_{i_j}\}_{j=0}^{\gamma}$, $\gamma \lnsim \delta$, $1 \doteq p(v_{i_j}) \lesssim 1 - 1/\delta$ can hold. But, since $\sum_{j=0}^{\gamma} \tilde{\delta}_2(\xi_{i_j}) = 0$, we get also $E_1\left(\sum_{i=0}^{\delta}(\xi_i)\right) = 0$. □

Now, we can turn our attention to limit theorems.

**Theorem 5** (The weak law of large numbers)**.** *Let $\{\xi_i\}_{i=1}^{\delta}$ be a sequence of independent random variables. Then*

**(a)** *if $\tilde{\delta}_1(\xi_i) = 0$ and $\sum_{i=1}^{\delta} \tilde{\delta}_2(\xi_i) \lnsim \delta^2$ for all $i \leq \delta$, then for all $\beta \sim \delta$, $\beta \leq \delta$ there holds* $p(1/\beta \sum_{i=1}^{\beta} \xi_i \doteq 0) \doteq 1$.

**(b)** *If $\tilde{\delta}_1(\xi_i)$ is bounded, $\sum_{i=1}^{\delta} \tilde{\delta}_2(\xi_i) \sim \delta^2$ and $\tilde{\delta}_2(\xi_i) \lessapprox \delta$, then $1/\delta \sum_{i=1}^{\delta} \xi_i$ is a standard random variable with a non-zero expected value of the 2-nd order, and if $\beta \simeq \delta$, $\beta \leq \delta$, then the probability distributions of $1/\delta \sum_{i=1}^{\delta} \xi_i$ and $1/\beta \sum_{j=1}^{\beta} \xi_{i_j}$ are infinitesimally shifted for any subsequence $\{\xi_{i_j}\}_{j=1}^{\beta}$.*

Proof: (a) Fix a $\beta \sim \delta$, $\beta \leq \delta$. By Theorem 4, $E_2(\frac{1}{\beta} \sum_{i=1}^{\beta} \xi_i) = \frac{1}{\beta^2} \sum_{i=1}^{\beta} \tilde{\delta}_2(\xi_i)$.

Denote $\eta = \frac{1}{\beta} \sum_{i=1}^{\beta}(\xi_i)$. Then, by our definition, there exists a $\gamma \in \mathbb{N} \setminus \mathbb{FN}$ such that

$$E_2(\eta) = \sum_{\substack{x \in u \\ -\gamma \leq x \leq \gamma}} x^2 p(\eta = x) \geq \sum_{\substack{x \in u \setminus [-\epsilon;\epsilon] \\ -\gamma \leq x \leq \gamma}} x^2 p(\eta = x) \dot{\geq} \epsilon^2 p(|\eta| \geq \epsilon)$$

for any $\epsilon \neq 0$, and this implies the assertion (a).[3]

(b) Since the random variables $\xi_i - \tilde{\delta}_1(\xi_i)$ fulfil the assumptions of Theorem 4, we get that $1/\beta \sum_{j=1}^{\beta} \xi_{i_j}$ is a standard random variable with a non-zero expected value of the 2-nd order for any chosen subsequence. To prove the remainder of this assertion, it is enough to realize that $\frac{1}{\delta^2} \sum_{j=\beta+1}^{\delta} \tilde{\delta}_2(\xi_{i_j}) \doteq 0$, where the sum runs through all the remaining random variables. □

**Example 1** (*The Poisson probability distribution.*)**.** Suppose we have a sequence of independent random variables $\{\xi_i\}_{i=1}^{\delta}$ with the following properties, holding for each $i \leq \delta$

(a) $\tilde{\delta}_1 \xi_i = \delta/\gamma$
(b) $\tilde{\delta}_2 \xi_i = \delta^2/\gamma$
(c) $p(\xi_i = \delta) \simeq 1/\gamma$,

where $\gamma \sim \delta$. Properties (b) and (c) imply that there exists a $\beta \lnapprox \delta$ such that $p(\xi_i \notin [-\beta, \beta]) \simeq 1/\gamma$. Denote $\eta = \frac{1}{\delta} \sum_{i=1}^{\delta}(\xi_i)$. By Theorem 5b, we know that $\eta$ is

[3] By $a \dot{\geq} b$ for $a,\ b \in \mathbb{Q}$ we mean $a \doteq b$, or $a \geq b$.

a standard random variable. A straightforward computation gives

$$P(\eta \doteq i) = \operatorname{mon}\left(\binom{\delta}{i}\cdot\left(1-\frac{1}{\gamma}\right)^{\delta-i}\cdot\left(\frac{1}{\gamma}\right)^{i}\right) = \left(\operatorname{mon}\left(\frac{\delta}{\gamma}\right)\right)^{i}\cdot\frac{1}{i!}\cdot\exp(\delta/\gamma),$$

where $i \in \mathbb{FN}$, and this is the usual Poisson probability distribution with its parameter $\operatorname{mon}(\delta/\gamma)$.

**Theorem 6** (The strong law of large numbers.). *Let $\{\xi_i\}_{i=1}^{\delta}$ be a sequence of independent random variables with the following properties*

(a) *the random variables $\xi_i$, for $i \in \mathbb{FN}$, are standard*

(b) $\tilde{\delta}_1(\xi_i) = 0$ *for each $i \leq \delta$*

(c) $\displaystyle\sum_{i=\gamma}^{\delta}\frac{\tilde{\delta}_2(\xi_i)}{i^2} = 0$ *for each infinite $\gamma \leq \delta$.*

*Then there exists a class $\mathfrak{A}$ with $P(\mathfrak{A}) = 1$ such that for each $\omega \in \mathfrak{A}$ and each infinite $\gamma \leq \delta$ there holds*

$$\frac{1}{\gamma}\sum_{i=1}^{\gamma}\xi_i(\omega) \doteq 0.$$

Proof: Take an $\epsilon \not\doteq 0$ and a $\gamma \leq \delta$, and denote

$$a_k = \left\{\omega; \max_{\gamma<k}\left|\frac{1}{\gamma}\cdot\sum_{i=1}^{\gamma}\xi_i(\omega)\right| \leq \epsilon \quad \& \quad \left|\frac{1}{k}\cdot\sum_{i=1}^{k}\xi_i(\omega)\right| > \epsilon\right\}$$

and

$$\zeta_\beta = \sum_{i=\gamma}^{\beta}\frac{\xi_i}{i} + \frac{1}{\gamma}\sum_{i=1}^{\gamma-1}\xi_i.$$

Obviously, the events $a_k$ are mutually excluding, i.e. the sets $a_k$ are pairwise disjoint, hence $\sum_{k=\gamma}^{\delta} p(a_k) \leq 1$. Now, let us estimate $p(\{\omega; \max_{\gamma\leq i\leq\delta}|\frac{1}{i}\sum_{j=1}^{i}\xi_j(\omega)| > \epsilon\})$. Since the random variables $\xi_i$ are standard for all $i \in \mathbb{FN}$, we can take any infinite $\rho \in \mathbb{N}$ and the formula

$$p(\{\omega; (\forall i \in \mathbb{FN})(|\xi_i(\omega)| \geq \rho)\}) \doteq 0$$

holds. Hence, by Prolongation Axiom, there is an infinite $\tau$ such that

$$p(\{\omega; (\forall i \leq \tau)(|\xi_i(\omega)| \geq \rho)\}) \doteq 0 \tag{4}$$

holds. So, a straightforward computation gives

$$(5)\qquad p\left(\left\{\omega;\ \max_{\gamma\le i\le\delta}\left|\frac{1}{i}\sum_{j=1}^{i}\xi_j(\omega)\right|>\epsilon\right\}\right)\le p(\max_{\gamma\le i\le\delta}|\zeta_i|>\epsilon)=\sum_{i=\gamma}^{\delta}p(a_i).$$

From (4) we get

$$(6)\qquad \begin{aligned}\frac{\rho^2}{\gamma^2}\sum_{i=1}^{\tau}p(\xi_i\le\rho)+\sum_{i=\tau+1}^{\delta}\frac{1}{i^2}\cdot\tilde{\delta}_2(\xi_i)\ge E_2(\zeta_\delta)\ge\sum_{k=\gamma}^{\delta}\sum_{\omega\in a_k}\zeta_\delta^2 p(\{\omega\})\ge\\ \ge\epsilon^2\cdot\sum_{k=\gamma}^{\delta}p(a_k).\end{aligned}$$

Up to now, $\rho$ has been arbitrarily chosen, hence we can take it in such a way that $\rho\lnapprox\gamma$, and then, by (5), (6) and Property (c) we get

$$p\left(\left\{\omega;\ \max_{\gamma\le i\le\delta}\left|\frac{1}{i}\sum_{j=1}^{i}\xi_j(\omega)\right|>\epsilon\right\}\right)\doteq 0,$$

and this implies the assertion in question. □

**Example 2** (*The normal probability distribution.*)**.** Denote $\vartheta_i$ the random variables which achieve just two values 1 and $-1$ with $p(\vartheta_i=1)=1/2$ and $p(\vartheta_i=-1)=1/2$ and let $\{\vartheta_i\}_{i=1}^{\gamma}$ be a sequence of independent random variables. Further denote $\eta=\frac{1}{\sqrt{\gamma}}\sum_{i=1}^{\gamma}\vartheta_i$.[4] By Theorem 5b, we know that $\eta$ is a standard random variable, and, by Theorem 4, we know that $E_1(\eta)=0$ and $E_2(\eta)=1$. This probability distribution we will call normal. It is just an easy exercise to show that $\eta$ has its expected value of each finite order.

It is known that, if $\{\eta_i\}_{i=1}^{\delta}$ is a sequence of our normally distributed, independent, random variables and $\{\theta_i\}_{i=1}^{\delta}$ is a sequence of positive rational numbers, $\delta\lnapprox\gamma$, $\gamma$ being the very number from Example 2, then the probability distribution of the random variable $\frac{1}{\sqrt{\sum_{i=1}^{\delta}\theta_i^2}}\sum_{i=1}^{\delta}\theta_i\cdot\eta_i$ is infinitesimally shifted from the, already defined, normal probability distribution.

[4]By $\sqrt{\gamma}$ we will understand any number $\epsilon\in\mathbb{Q}$ such that $\epsilon^2=_\gamma\gamma$. In the next theorem we will use the symbol $\sqrt{\gamma}$ also for $\delta$-numbers. In that case by $\sqrt{\gamma}$ we will understand an $\epsilon\in\mathbb{Q}$ such that $\epsilon^2=_\delta\gamma$.

**Theorem 7** (The central limit theorem.). *Let $\{\xi_i\}_{i=1}^{\delta}$ be a sequence of independent random variables with the following properties, holding for each $i \leq \delta$*

(a) $\tilde{\delta}_1 \xi_i = 0,$

(b) $0 \neq \tilde{\delta}_2 \xi_i \lesssim S^2/\delta,$

(c) $\displaystyle\sum_{i=1}^{\delta} p(|\xi_i| \geq S/\gamma) \doteq 0 \quad \text{for a} \quad \gamma \gtrapprox \sqrt[3]{\delta},$

*where $S^2 = \displaystyle\sum_{i=1}^{\delta} \tilde{\delta}_2 \xi_i$ and $S = \sqrt{S^2}$. Further, let $\{\eta_i\}_{i=1}^{\delta}$ be a sequence of independent, normally distributed random variables. Then the probability distributions of $\displaystyle\frac{1}{S}\sum_{i=1}^{\delta} \xi_i$ and $\displaystyle\frac{1}{S}\sum_{i=1}^{\delta} \sqrt{\tilde{\delta}_2(\xi_i)} \cdot \eta_i$ are infinitesimally shifted.*

Proof: Denote $\zeta_i = \sqrt{\tilde{\delta}_2(\xi_i)} \cdot \eta_i$. By Theorem 2 we know that, in order to prove our assertion, it is enough to prove that the expected values of each finite order of the random variables $\displaystyle\frac{1}{S}\sum_{i=1}^{\delta} \xi_i$ and $\displaystyle\frac{1}{S}\sum_{i=1}^{\delta} \zeta_i$ equal each other. By Theorem 4 and Properties (a-c), their expected values of the first and second orders equal each other. So, by Lemma 2 and Property (c), since we know that a normally distributed random variable has expected values of all finite orders, it is enough to prove that for each $n \in \mathbb{FN}$, $n \geq 3$

$$(7) \qquad 0 \doteq \sum_{\omega \in \bar{\Omega}} \left( \left( \frac{1}{S} \sum_{i=1}^{\delta} \xi_i(\omega) \right)^n p(\{\omega\}) - \left( \frac{1}{S} \sum_{i=1}^{\delta} \zeta_i(\omega) \right)^n p(\{\omega\}) \right),$$

where $\bar{\Omega}$ is a subset of the common probability space of the random variables $\xi_i$ and $\zeta_i$, $\bar{\Omega} \subseteq \prod_{i=1}^{\delta} \Omega_i \times \prod_{i=1}^{\delta} \Omega'_i$, such that for each $i \leq \delta$ and each $\omega \in \bar{\Omega}$ $|\xi_i(\omega)| \leq S/\gamma$, $|\zeta_i(\omega)| \leq S/\gamma$ holds. Suppose, we have already proved Formula (7) for all $n < m$. So, let us prove it for $m$. Denote $\theta_i = \displaystyle\sum_{j=1}^{i-1} \xi_j + \sum_{j=i+1}^{\delta} \zeta_i$. Obviously, it is enough to prove that for each $i \leq \delta$

$$(8) \quad 0 =_\delta \sum_{\omega \in \bar{\Omega}} \left( \left( \frac{1}{S}(\theta_i(\omega) + \xi_i(\omega)) \right)^m p(\{\omega\}) - \left( \frac{1}{S}(\theta_i(\omega) + \zeta_i(\omega)) \right)^m p(\{\omega\}) \right)$$

holds. And, of course, we can assume Formula (8) to be already proved for all $m < n$. Because of Properties (b) and (c), without loss of generality, we can assume for each $i \leq \delta \quad p(|\xi_i| > S/\gamma) =_\delta 0 \quad$ and $\quad p(|\zeta_i| > S/\gamma) =_\delta 0$. Hence,

$$(9) \quad \sum_{\omega \in \bar{\Omega}} \left( \left( \frac{1}{S}(\theta_i(\omega) + \xi_i(\omega)) \right)^m p(\{\omega\}) - \left( \frac{1}{S}(\theta_i(\omega) + \zeta_i(\omega)) \right)^m p(\{\omega\}) \right) =_\delta$$

$$=_\delta \frac{1}{S^m} \cdot \sum_{\omega \in \bar{\Omega}} p(\{\omega\}) \cdot \Bigg( (\theta_i(\omega)^m - \theta_i(\omega)^m) + \binom{m}{1} \theta_i(\omega)^{m-1} (\xi_i(\omega) - \zeta_i(\omega)) +$$

$$+ \binom{m}{2} \theta_i(\omega)^{m-2} (\xi_i(\omega)^2 - \zeta_i(\omega)^2) + \cdots + \xi_i(\omega)^m - \zeta_i(\omega)^m \Bigg).$$

Since the random variables $\theta_i, \quad \xi_i, \quad \zeta_i$ are mutually independent and since Formula (8) has already been proved for all $m < n$, we get

$$\frac{1}{S^m} \cdot \sum_{\omega \in \bar{\Omega}} p(\{\omega\}) \cdot \binom{m}{k} \theta_i(\omega)^{m-k} (\xi_i(\omega)^k - \zeta_i(\omega)^k) =_\delta$$

$$=_\delta \frac{1}{S^m} \binom{m}{k} \sum_{\omega \in \bar{\Omega}} \theta_i(\omega)^{m-k} p(\{\omega\}) \cdot \sum_{\omega \in \bar{\Omega}} (\xi_i(\omega)^k - \zeta_i(\omega)^k) p(\{\omega\}) =_\delta 0,$$

and hence the right-hand side of Formula (9) is $\delta$-equal to

$$\sum_{\omega \in \bar{\Omega}} \frac{\xi_i(\omega)^m - \zeta_i(\omega)^m}{S^m} \cdot p(\{\omega\}).$$

Because of Property (c) and our assumption $m \geq 3$,

$$\sum_{\omega \in \bar{\Omega}} \frac{|\xi_i(\omega)|^m}{S^m} p(\{\omega\}) =_\delta \sum_{\omega \in \bar{\Omega}} \frac{|\zeta_i(\omega)|^m}{S^m} p(\{\omega\}) \leq_\delta \frac{S^m}{\gamma^m \cdot S^m} =_\delta \frac{1}{\gamma^m} =_\delta 0$$

holds, and this implies our assertion. $\square$

## References


[K-Z] Kalina M., Zlatoš P., *Borel classes in AST. Measurability, cuts and equivalence*, Comment. Math. Univ. Carolinae **30** (1989), 357–372.

[L] Loeb P.A., *Conversion from nonstandard to standard measure spaces and applications in probability theory*, Trans. AMS **211** (1975), 113–122.

[N] Nelson E., *Radically Elementary Probability Theory*, Princeton University Press, Princeton, 1987.

[V] Vopěnka P., *Mathematics in the Alternative Set Theory*, Teubner Texte, Leipzig, 1979.



Department of Mathematics, Slovak Technical University, Radlinského 11, 813 68 Bratislava, Slovakia

*E-mail*: kalina@cvt.stuba.sk



(*Received* April 26, 1993)