Notes on approximation in the Musielak-Orlicz spaces of vector multifunctions

Andrzej Kasperski

Abstract. We introduce the spaces $M^1_{Y,\varphi}$, $M^{o,n}_{Y,\varphi}$, $\tilde{M}^o_{Y,\varphi}$ and $M^o_{Y,\mathbf{d},\varphi}$ of multifunctions. We prove that the spaces $M^1_{Y,\varphi}$ and $M^o_{Y,\mathbf{d},\varphi}$ are complete. Also, we get some convergence theorems.

Keywords: Musielak-Orlicz space, multifunction, modular space of multifunctions, integral operator, modular approximation

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1. Introduction

In this paper we extend the results of [2] and [3] to the case of the spaces $M^1_{Y,\varphi}$, $\tilde{M}^o_{Y,\varphi}$ and $M^o_{Y,\mathbf{d},\varphi}$ of multifunctions. All definitions and theorems connected with the idea of Musielak-Orlicz space can be found in [4] and [5].

Let I be a bounded interval. Let (I, Σ, μ) be the Lebesgue measure space. Let X be a real separable Hilbert space with the norm $\| \circ \|_X$. We denote by $L^\varphi(I, X)$ the Musielak-Orlicz space of all strongly measurable functions $x : I \to X$ generated by a modular

$$\varrho(x) = \int_I \varphi(t, \|x(t)\|_X) d\mu,$$

where φ is a φ-function with a parameter such that $\varphi : I \times R \to R_+, \varphi(t, \circ)$ is an even continuous function, nondecreasing for $u \geq 0$, $\varphi(t, u) = 0$ iff $u = 0$ for every $t \in I$, $\varphi(\circ, u)$ is measurable for every $u \in R$ and $\lim_{u\to\infty} \varphi(t, u) = \infty$ for a.e. $t \in I$. The space $L^\varphi(I, X)$ is N-complete (see [5, Corollaries 3.3]).

Let $\mathbf{N}$ be the set of all positive integers.

2. Completeness

Let Y be a real separable Hilbert space. Let o denote the zero element in Y. Let

$$\operatorname{dist}(A, B) = \max(\sup_{x\in A} \inf_{y\in B} \|x - y\|_Y, \sup_{y\in B} \inf_{x\in A} \|x - y\|_Y),$$

for all nonempty bounded $A, B \subset Y$. Let

$$M_Y(I) = \{F : I \to 2^Y : F(s) \text{ is nonempty for every } s \in I, \text{ closed and bounded for a.e. } s \in I\}.$$

For $F,G \in M_Y(I)$ we introduce the function $\mathbf{d}(F,G)$ by the formula:

$$\mathbf{d}(F,G)(t) = \begin{cases} 0, & \text{if } F(t) = G(t) \\ \operatorname{dist}(F(t),G(t)), & \text{if } F(t), G(t) \text{ are bounded} \\ \infty, & \text{if } F(t) \neq G(t) \text{ and } F(t) \text{ or } G(t) \text{ is unbounded} \end{cases}$$

for every $t \in I$.

Remark 1. If X is a Banach space, then the space of all nonempty closed and bounded subsets of X with dist is a complete metric space.

Lemma 1. *Let $F_n \in M_Y(I)$ for every $n \in \mathbf{N}$. If:*

(a) *there is $n_o > 0$ such that $\mathbf{d}(F_n, F_m)$ are measurable for $m, n > n_o$,*
(b) *for every $\varepsilon > 0$ and every $\delta > 0$ there exists $K > n_o$ such that $\mu(\{t \in I : \mathbf{d}(F_n, F_m)(t) \geq \delta\}) < \varepsilon$, for all $m, n > K$,*

then there exist a subsequence $\{F_{n_k}\}$ of the sequence $\{F_n\}$ and $F \in M_Y(I)$ such that $\mathbf{d}(F_{n_k}, F) \to 0$ a.e. and $\mathbf{d}(F_n, F)$ are measurable for $n > n_o$.

Proof: Let $F_n \in M_Y(I)$ for every $n \in \mathbf{N}$. We have from the assumptions that there exists $N(k)$ such that $\mu(\{t \in I : \mathbf{d}(F_n, F_m)(t) \geq 2^{-k}\}) < 2^{-k}$ for all $m, n > N(k)$. Let $n_1 = N(1)$, $n_2 = \max\{N(2), N(1)+1\}, \ldots, n_m = \max\{N(m), N(m-1)+1\}$. Let $\varepsilon > 0$ be arbitrary. So there is i_0 such that $2^{i_0 - 1} < \varepsilon$. Let $i_0 < i < j$. Let $A_i = \{t \in I : \mathbf{d}(F_{n_{i+1}}, F_{n_i})(t) \geq 2^{-i}\}$. It is easy to see that $\mu(\bigcup_{i=i_0}^{\infty} A_i) < \varepsilon$ and for $t \in I \setminus \bigcup_{i=i_0}^{\infty} A_i$ we have

$$\mathbf{d}(F_{n_j}, F_{n_i})(t) \leq \sum_{k=i}^{j-1} \mathbf{d}(F_{n_{k+1}}, F_{n_k})(t) \leq \sum_{k=i}^{\infty} \mathbf{d}(F_{n_{k+1}}, F_{n_k})(t) < \varepsilon.$$

So for the subsequence $\{F_{n_k}\}$ we have that for a.e. $t \in I$ and for every $\varepsilon > 0$ there is $K > 0$ such that $\mathbf{d}(F_{n_k}, F_{n_l})(t) < \varepsilon$ for all $k, l > K$. Hence by Remark 1 there is $F \in M_Y(I)$ such that $\mathbf{d}(F_{n_k}, F) \to 0$ as $k \to \infty$ a.e. and $\mathbf{d}(F_n, F)$ are measurable for $n > n_0$ because $\mathbf{d}(F_n, F) = \lim_{k \to \infty} \mathbf{d}(F_{n_k}, F_n)$ a.e.

Let:

$M(I, Y) = \{x : I \to Y : x \text{ is strongly measurable}\}$,

$M(I, R) = \{q : I \to R : q \text{ is measurable}\}$.

We denote for all $a \in Y$, R, $r \geq 0$, $B(a, r) = \{x \in Y : \|x - a\|_Y \leq r\}$,

$R(o,r,\mathrm{R}) = \{x \in Y : r \le \|x\|_Y \le \mathrm{R}\}$. Let:

$$\begin{aligned} M_Y^{o,n}(I) = \{&F \in M_Y(I) : F(s) = \bigcup_{i=1}^{n} R(o, r_F^i(s), R_F^i(s)) \text{ for every } s \in I, r_F^i(\circ),\\ &R_F^i(\circ) \in M(I,R) \text{ for } i = 1, \dots, n,\ R_F^i(t) \le r_F^{i+1}(t) \text{ for } t \in I,\\ &i = 1, \dots, n-1, \text{ if } n > 1\},\end{aligned}$$

$$\begin{aligned} \tilde{M}_Y^o(I) &= \bigcup_{i=1}^{\infty} M_Y^{o,i}(I),\\ M_Y^o(I) &= \{F \in M_Y(I) : F(s) = B(o, R_F(s)) \text{ for every } s \in I, R_F(\circ) \in M(I,R)\},\\ M_Y^1(I) &= \{F \in M_Y(I) : F(s) = B(a_F(s), r_F(s)) \text{ for every } s \in I, a_F(\circ) \in\\ &\qquad M(I,Y), r_F(\circ) \in M(I,R)\}.\end{aligned}$$

If $F, G \in M_Y^1(I)$ and $F(t) = G(t)$ for a.e. $t \in I$, then $F = G$ in $M_Y^1(I)$. If $F, G \in \tilde{M}_Y^o(I)$ and $F(t) = G(t)$ for a.e. $t \in I$, then $F = G$ in $\tilde{M}_Y^o(I)$. In the set $M_Y^1(I)$ we introduce the operations $\odot : R \times M_Y^1(I) \to M_Y^1(I)$, $\oplus : M_Y^1(I) \times M_Y^1(I) \to M_Y^1(I)$ as follows: let $F_1, F_2 \in M_Y^1(I)$, $a \in R$, $F_1(s) = B(a_{F_1}(s), r_{F_1}(s))$, $F_2(s) = B(a_{F_2}(s), r_{F_2}(s))$ for every $s \in I$; if $F = F_1 \oplus F_2$ then

$$\begin{aligned} F(s) = B(a_{F_1}(s) + a_{F_2}(s), r_{F_1}(s) + r_{F_2}(s)) \quad \text{for every} \quad s \in I,\\ \text{if} \quad G = a \odot F_1, \quad \text{then} \quad G(s) = B(a a_{F_1}(s), a r_{F_1}(s)) \quad \text{for every} \quad s \in I.\end{aligned}$$

It is easy to see that $F, G \in M_Y^1(I)$. In the set $\tilde{M}_Y^o(I)$ we introduce the operations $\odot : R \times \tilde{M}_Y^o(I) \to \tilde{M}_Y^o(I)$, $\oplus : \tilde{M}_Y^o(I) \times \tilde{M}_Y^o(I) \to \tilde{M}_Y^o(I)$ as follows: let $F_1, F_2 \in \tilde{M}_Y^o(I)$, $a \in R$,

$$F_1(s) = \bigcup_{i=1}^{n} R(o, r_{F_1}^i(s), R_{F_1}^i(s)),\ F_2(s) = \bigcup_{i=1}^{m} R(o, r_{F_2}^i(s), R_{F_2}^i(s)) \text{ for all } s \in I,$$

$$\text{if } F = F_1 \oplus F_2, \text{ then } F(s) = \bigcup_{\substack{1 \le i \le n\\ 1 \le j \le m}} R(o, r_{F_1}^i(s) + r_{F_2}^j(s), R_{F_1}^i(s) + R_{F_2}^j(s))$$

for every $s \in I$, if

$$G = a \odot F_1, \text{ then } G(s) = \bigcup_{i=1}^{n} R(o, a r_{F_1}^i(s), a R_{F_1}^i(s))$$

for every $s \in I$. It is easy to see that $F, G \in \tilde{M}_Y^o(I)$. $\square$

Let now

$$\begin{aligned} M^o_{Y,\varphi}(I) &= \{F \in M^o_Y(I) : r_F(\circ) \in L^\varphi(I,R)\},\\ M^1_{Y,\varphi}(I) &= \{F \in M^1_Y(I) : a_F(\circ) \in L^\varphi(I,Y), r_F(\circ) \in L^\varphi(I,R)\},\\ \tilde{M}^o_{Y,\varphi}(I) &= \{F \in \tilde{M}^o_Y(I) : r^i_F(\circ), R^i_F(\circ) \in L^\varphi(I,R) \text{ for } i=1,\dots,n,\\ &\qquad \text{if } F \in M^{o,n}_Y(I)\}. \end{aligned}$$

Remark 2. If $F, G \in M^1_{Y,\varphi}(I)$, then $\mathbf{d}(F,G)$ is measurable.

Proof: It is easy to see that

$$\mathbf{d}(F,G)(s) = \|a_F(s) - a_G(s)\|_Y + \mid r_F(s) - r_G(s) \mid \text{ for a.e. } s \in I,$$

so $\mathbf{d}(F,G)$ is measurable. □

Remark 2'. If $F, G \in \tilde{M}^o_{Y,\varphi}(I)$, then $\mathbf{d}(F,G)$ is measurable.

Proof: Let

$$F(s) = \bigcup_{i=1}^{n} R(o, r^i_F(s), R^i_F(s)), G(s) = \bigcup_{j=1}^{m} R(o, r^j_G(s), R^j_G(s))$$

for $s \in I$. It is easy to see that

$$\mathbf{d}(F,G)(s) = \operatorname{dist}(\bigcup_{i=1}^{n} [r^i_F(s), R^i_F(s)], \bigcup_{j=1}^{m} [r^j_G(s), R^j_G(s)]) \text{ for a.e. } s \in I,$$

so $\mathbf{d}(F,G)$ is measurable (see [1, Remark 1, p. 120]). □

Definition 1. Let $F, F_n \in M_Y(I)$ for every $n \in \mathbf{N}$. We write $F_n \xrightarrow{d,\varphi} F$, if there exists $n_o > 0$ such that $\mathbf{d}(F_n, F)$ are measurable for $n > n_o$ and

$$\int_I \varphi(t, a\mathbf{d}(F_n, F)(t))\, dt \to 0 \text{ as } n \to \infty \text{ for every } a > 0.$$

Definition 2. Let $F_n \in M_Y(I)$ for every $n \in \mathbf{N}$. We say that the sequence $\{F_n\}$ fulfils the $(C, \mathbf{d}, \varphi)$-condition, if there exists $n_o > 0$ such that $\mathbf{d}(F_n, F_m)$ are measurable for $n, m > n_o$ and for every $\varepsilon > 0$ and every $a > 0$ there is $K > n_o$ such that $\int_I \varphi(t, a\mathbf{d}(F_n, F_m)(t))\, dt < \varepsilon$ for all $m, n > K$.

Definition 3. Let $A \subset M_Y(I)$. We say that A is $(C, \mathbf{d}, \varphi)$-complete, if for every sequence $\{F_n\}$ such that $F_n \subset A$ for every $n \in \mathbf{N}$ and the sequence $\{F_n\}$ fulfils the $(C, \mathbf{d}, \varphi)$-condition, there is $F \in A$ such that $F_n \xrightarrow{d,\varphi} F$.

Theorem 1. *$M^1_{Y,\varphi}(I)$ is $(C,\mathbf{d},\varphi)$-complete.*

Proof: Let $F_n \in M^1_{Y,\varphi}(I)$ for every $n \in \mathbf{N}$ and let the sequence $\{F_n\}$ fulfil the $(C,\mathbf{d},\varphi)$-condition. Let $F_n(s) = B(a_{F_n}(s), r_{F_n}(s))$ for every $s \in I$ and every $n \in \mathbf{N}$. Then $\{a_{F_n}\}$ is a Cauchy sequence in the Musielak-Orlicz space $L^\varphi(I,Y)$ and $\{r_{F_n}\}$ is a Cauchy sequence in the Musielak-Orlicz space $L^\varphi(I,R)$. So there are $\mathbf{a} \in L^\varphi(I,Y)$ and $\mathbf{r} \in L^\varphi(I,R)$ such that

$$\varrho(a(\mathbf{a} - a_{F_n})) \to 0, \varrho(a(\mathbf{r} - r_{F_n})) \to 0 \text{ as } n \to \infty \text{ for every } a > 0.$$

Let $\mathbf{F}(s) = B(\mathbf{a}(s), \mathbf{r}(s))$ for every $s \in I$. It is easy to see that $\mathbf{F} \in M^1_{Y,\varphi}(I)$ and $F_n \xrightarrow{d,\varphi} \mathbf{F}$. □

Remark 3. $\tilde{M}^o_{Y,\varphi}(I)$ is not $(C,\mathbf{d},\varphi)$-complete.

Now, let us denote

$$M^o_{Y,\mathbf{d}}(I) = \{F \in M_Y(I) : \mathbf{d}(F_n, F) \to 0 \text{ a.e. for some } F_n \in \tilde{M}^o_{Y,\varphi}(I), n \in \mathbf{N}\},$$

$$M^o_{Y,\mathbf{d},\varphi}(I) = \{F \in M^o_{Y,\mathbf{d}}(I) : F_n \xrightarrow{d,\varphi} F \text{ for some } F_n \in \tilde{M}^o_{Y,\varphi}(I), n \in \mathbf{N}\}.$$

Remark 4. If $F, G \in M^o_{Y,\mathbf{d}}(I)$, then $\mathbf{d}(F,G)$ is measurable.

Proof: Let $F, G \in M^o_{Y,\mathbf{d}}(I)$. So there are $F_n, G_n \in \tilde{M}^o_{Y,\varphi}(I)$, $n \in \mathbf{N}$ such that $\mathbf{d}(F_n, F) \to 0$ and $\mathbf{d}(G_n, G) \to 0$ as $n \to \infty$ a.e. So $\mathbf{d}(F_n, G_n) \to \mathbf{d}(F,G)$ as $n \to \infty$ a.e. Hence $\mathbf{d}(F,G)$ is measurable because from Remark 2' $\mathbf{d}(F_n, G_n)$ are measurable for $n \in \mathbf{N}$. □

Theorem 2. *$M^o_{Y,\mathbf{d},\varphi}(I)$ is $(C,\mathbf{d},\varphi)$-complete.*

Proof: Let $F_n \in M^o_{Y,\mathbf{d},\varphi}(I)$ for every $n \in \mathbf{N}$, and let the sequence $\{F_n\}$ fulfil the $(C,\mathbf{d},\varphi)$-condition. It is easy to prove that the sequence $\{F_n\}$ fulfils the assumptions of Lemma 1, so there exist a subsequence $\{F_{n_k}\}$ of the sequence $\{F_n\}$ and $F \in M_Y(I)$ such that $\mathbf{d}(F_{n_k}, F) \to 0$ a.e. and $\mathbf{d}(F_n, F)$ are measurable. We have by Fatou Lemma

$$\int_I \varphi(t, a\mathbf{d}(F_n, F)(t))\, dt \le \varepsilon \text{ for } n > K,$$

so $F_n \xrightarrow{d,\varphi} F$. For every $n \in \mathbf{N}$, $\varepsilon > 0$, $a > 0$ there exists $F^n_n \in \tilde{M}^o_{Y,\varphi}(I)$ such that $\int_I \varphi(t, a\mathbf{d}(F^n_n, F_n)(t))\, dt < \varepsilon$, so we have

$$\int_I \varphi(t, \frac{a}{2}\mathbf{d}(F^n_n, F)(t))\, dt \le$$

$$\le \int_I \varphi(t, a\mathbf{d}(F^n_n, F_n)(t))\, dt + \int_I \varphi(t, a\mathbf{d}(F_n, F)(t))\, dt < 2\varepsilon$$

for $n > K$, hence $F \in M^o_{Y,\mathbf{d},\varphi}(I)$ and $M^o_{Y,\mathbf{d},\varphi}(I)$ is $(C,\mathbf{d},\varphi)$-complete. □

The spaces $M^1_{Y,\varphi}(I)$ and $M^o_{Y,\mathbf{d},\varphi}(I)$ will be called the Musielak-Orlicz spaces of vector multifunctions.

3. On the operator H

Let $H{:}I \times Y \to Y$ and let

$$\mathbf{H}(F)(t) = \{H(t,x) : x \in F(t)\} \text{ for every } t \in I, F \in M_Y(I).$$

Lemma 2. *Let the function H fulfil the following conditions:*

(a) *$H(s,x)$ is a strongly measurable function as a function of s for every $x \in Y$,*
(b) *there exists $L > 0$ such that $\|H(s,x) - H(s,y)\|_Y \le L\|x-y\|_Y$ for all $s \in I$, $x, y \in Y$,*
(c) *$H(s,o) = o$ for every $s \in I$,*
(d) *if $\|x\|_Y < \|y\|_Y$, then $\|H(s,x)\|_Y < \|H(s,y)\|_Y$ and if $\|x\|_Y = \|y\|_Y$, then $\|H(s,x)\|_Y = \|H(s,y)\|_Y$ for every $s \in I$,*
(e) *for every $t \in I$ and every $y \in Y$ there is $x \in Y$ such that $y = H(t,x)$.*

Then $\mathbf{H} : M^o_{Y,\varphi}(I) \to M^o_{Y,\varphi}(I)$ and $\mathbf{H} : \tilde{M}^o_{Y,\varphi}(I) \to \tilde{M}^o_{Y,\varphi}(I)$.

Proof: We only prove that $\mathbf{H} : M^o_{Y,\varphi}(I) \to M^o_{Y,\varphi}(I)$. The proof that $\mathbf{H} : \tilde{M}^o_{Y,\varphi}(I) \to \tilde{M}^o_{Y,\varphi}(I)$ as analogous is omitted. Let $F \in M^o_{Y,\varphi}(I)$. We prove that there exists $r_{\mathbf{H}(F)} \in L^\varphi(I,R), r_{\mathbf{H}(F)}(t) \ge 0$ for every $t \in I$, such that $\mathbf{H}(F)(t) = B(o, r_{\mathbf{H}(F)}(t))$ for every $t \in I$. Let $x \in Y$, $x \ne o$ be arbitrary. Let now $\xi(t) = x r_F(t)/\|x\|_Y$ for every $t \in I$. It is easy to see that $\xi \in M(I,Y) \cap F$ and $\|\xi(t)\|_Y = r_F(t)$ for every $t \in I$. Let $r_{\mathbf{H}(F)}(t) = \|H(t,\xi(t))\|_Y$ for every $t \in I$. We have

$$\sup_{z \in \mathbf{H}(F)(t)} \|z\|_Y = \sup_{x \in F(t)} \|H(t,x)\|_Y \le \|H(t,\xi(t))\|_Y,$$

for every $t \in I$, so $\mathbf{H}(F)(t) \subset B(o, r_{\mathbf{H}(F)}(t))$ for every $t \in I$. For every $a > 0$ we have

$$\begin{aligned}\int_I \varphi(t, a r_{\mathbf{H}(F)}(t))\,dt &= \int_I \varphi(t, a\|H(t,\xi(t))\|_Y)\,dt \le \int_I \varphi(t, aL\|\xi(t)\|_Y)\,dt \\ &= \int_I \varphi(t, aLr_F(t))\,dt.\end{aligned}$$

So $r_{\mathbf{H}(F)} \in L^\varphi(I,R)$. Let $t \in I$ be arbitrary, let $y \in B(o, r_{\mathbf{H}(F)}(t))$.
From (e) we obtain that there exists $\overline{x} \in Y$ such that $y = H(t,\overline{x})$. So $\|H(t,\overline{x})\|_Y \le \|H(t,\xi(t))\|_Y$. Hence from (d) we obtain that $\|\overline{x}\|_Y \le r_F(t)$. So $\overline{x} \in F(t)$ and $y \in \mathbf{H}(F)(t)$. Hence $\mathbf{H}(F)(t) = B(o, r_{\mathbf{H}(F)}(t))$ for every $t \in I$. □

Remark 5. Let $\mathcal{C}(F)(t) = \mathbf{H}(F + (-a_F))(t)$ for every $t \in I$, where $F(t) = B(a_F(t), r_F(t))$ for every $t \in I$. If the assumptions of Lemma 2 hold, then

$$\mathcal{C} : M^1_{Y,\varphi}(I) \to M^o_{Y,\varphi}(I).$$

Remark 6. Let the assumptions of Lemma 2 hold. If

(i) $H(s,A)$ is closed for every nonempty and closed $A \subset Y$ and for a.e. $s \in I$, then $\mathbf{H} : M^o_{Y,\mathbf{d},\varphi}(I) \to M^o_{Y,\mathbf{d},\varphi}(I)$.

Proof: The proof is analogous to that of Theorem 1' in [2] so we give only the sketch of it. First, from the assumptions (b), (c) of Lemma 2 and from the assumption (i) $\mathbf{H} : M_Y(I) \to M_Y(I)$. Second, from the assumption (b) of Lemma 2 we obtain that

$$\text{dist}\,(\mathbf{H}(F)(t), \mathbf{H}(G)(t)) \le L\,\text{dist}\,(F(t), G(t)) \tag{1}$$

for all $F, G \in M_Y(I)$ and $t \in I$ such that $F(t), G(t)$ are nonempty, bounded and closed. Third, from (1) and Lemma 2 we obtain that $\varrho(a\mathbf{d}(\mathbf{H}(F_n), \mathbf{H}(F))) \to 0$ as $n \to \infty$ for every $a > 0$, where $F \in M^o_{Y,\mathbf{d},\varphi}(I)$, $F_n \in \tilde{M}^o_{Y,\varphi}$, $n \in \mathbf{N}$ and $F_n \xrightarrow{d,\varphi} F$. So $\mathbf{H}(F) \in M^o_{Y,\mathbf{d},\varphi}(I)$ because from Lemma 2 $\mathbf{H}(F_n) \in \tilde{M}^o_{Y,\varphi}(I)$ for every $n \in \mathbf{N}$. $\square$

4. On the operators T'_v and T''_v

Let $\mathbf{V}$ be an abstract set of indices and let $\mathcal{V}$ be a filter of subsets of $\mathbf{V}$.

Definition 4. A function $g : \mathbf{V} \to R$ tends to zero with respect to $\mathcal{V}$, written $g(v) \xrightarrow{\mathcal{V}} 0$, if for every $\varepsilon > 0$ there is a set $V \in \mathcal{V}$ such that $| g(v) | < \varepsilon$ for every $v \in V$.

Definition 5. Let $F_v \in M_Y(I)$ for every $v \in \mathbf{V}$ and let $F \in M_Y(I)$. We write $F_v \xrightarrow{d,\varphi,\mathcal{V}} F$, if there is $V_o \in \mathcal{V}$ such that $\mathbf{d}(F_v, F)$ are measurable for every $v \in V_o$ and for every $\varepsilon > 0$, every $a > 0$ there is $V \in \mathcal{V}$ such that

$$\int_I \varphi(t, a\mathbf{d}(F_v, F)(t))\,dt < \varepsilon \text{ for every } v \in V_o \cap V.$$

Definition 6. Let $M(I) \subset M_Y(I)$. The family $T = (T_v)_{v\in\mathbf{V}}$ of operators, $T_v : M(I) \to M(I)$ for every $v \in \mathbf{V}$ will be called $(\mathbf{d}, \mathcal{V}, M(I))$-bounded, if there exist positive constants k_1, k_2 and a function $g : \mathbf{V} \to R_+$ such that $g(v) \xrightarrow{\mathcal{V}} 0$, and for all $F, G \in M(I)$ such that $\mathbf{d}(F,G)$ is measurable there exists a set $V_{F,G} \in \mathcal{V}$ such that $\mathbf{d}(T_v(F), T_v(G))$ are measurable and

$$\int_I \varphi(t, a\mathbf{d}(T_v(F), T_v(G))(t))\,dt \le k_1 \int_I \varphi(t, ak_2\mathbf{d}(F,G)(t))\,dt + g(v)$$

for every $a > 0$ and all $v \in V_{F,G}$.

Remark 7. Let the family T be $(\mathbf{d}, \mathcal{V}, M^o_{Y,\mathbf{d},\varphi}(I))$-bounded. If $T_v(F) \overset{d,\varphi,\mathcal{V}}{\longrightarrow} F$ for every $F \in \tilde{M}^o_{Y,\varphi}(I)$, then $T_v(F) \overset{d,\varphi,\mathcal{V}}{\longrightarrow} F$ for every $F \in M^o_{Y,\mathbf{d},\varphi}(I)$.

Proof: Let $a, \varepsilon > 0$ be arbitrary. Let $F \in M^o_{Y,\mathbf{d},\varphi}(I)$ be arbitrary. Let $G \in \tilde{M}^o_{Y,\varphi}$ and $V \in \mathcal{V}$ be such that $\varrho(3a\mathbf{d}(G,F)) < \frac{\varepsilon}{4}$, $\varrho(3ak_2\mathbf{d}(G,F)) < \frac{\varepsilon}{4k_1}$, $\varrho(3a\mathbf{d}(T_v(G),G)) < \frac{\varepsilon}{4}$, $g(v) < \frac{\varepsilon}{4}$ for every $v \in V$, where we may assume that $k_1 \geq 1$. It is easy to see that such G, V exist. We have for every $v \in V \cap V_{F,G}$

$$\begin{aligned}&\varrho(a\mathbf{d}(T_v(F),F)) \leq \\ &\qquad \leq \varrho(3a\mathbf{d}(T_v(F),T_v(G))) + \varrho(3a\mathbf{d}(T_v(G),G)) + \varrho(3a\mathbf{d}(G,F)) < \varepsilon.\end{aligned}$$

□

Let now $I = [0,b)$ and let us extend φ b-periodically to the whole R.

Definition 7. We shall say that the function φ is τ-bounded, if there are positive constants k_1, k_2 such that

$$\varphi(t-v,u) \leq k_1\varphi(t,k_2u) + f(t,v) \text{ for all } u,v,t \in R,$$

where $f : R \times R \to R_+$ is measurable and b-periodic with respect to the first variable and such that writing $h(v) = \int_0^b f(t,v)\,dt$ for every $v \in R$, we have $M = \sup_{v\in R} h(v) < \infty$ and $h(v) \to 0$ as $v \to 0$ or $v \to b$.

Let now $K_v : [0,b) \to R_+$ for every $v \in \mathbf{V}$ be integrable in $[0,b)$ and singular, i.e.

$$\sigma(v) = \int_0^b K_v(t)\,dt \overset{\mathcal{V}}{\longrightarrow} 1, \quad \sigma_\delta(v) = \int_\delta^{b-\delta} K_v(t)\,dt \overset{\mathcal{V}}{\longrightarrow} 0$$

for every $0 < \delta < \frac{b}{2}$, $\sigma = \sup_{v\in\mathbf{V}} \sigma(v) < \infty$. Let us extend K_v b-periodically to the whole R.

Let $q : [0,b) \to R$ be measurable and let us extend q b-periodically to the whole R. We introduce the family of operators $A^1 = (A^1_v)_{v\in\mathbf{V}}$ by the formula:

$$A^1_v(q)(t) = \int_0^b K_v(s-t)q(s)\,ds$$

for every $v \in \mathbf{V}$ and every $t \in [0,b)$.

Let $x : [0,b) \to Y$ be strongly measurable and let us extend x b-periodically to the whole R. We introduce the family of operators $A^2 = (A^2_v)_{v\in\mathbf{V}}$ by the formula:

$$A^2_v(x)(t) = \begin{cases} \int_0^b K_v(s-t)x(s)\,ds, & \text{if } \int_0^b K_v(s-t)\|x(s)\|_Y\,ds < \infty \\ o, & \text{if } \int_0^b K_v(s-t)\|x(s)\|_Y\,ds = \infty \end{cases}$$

for every $v \in \mathbf{V}$ and every $t \in [0,b)$.

Let us extend F b-periodically to the whole R.

Let $\mathcal{B}_v(F) = \{A^2_v(x) : x \in M([0,b),Y) \cap F\}$ for every $F \in M_Y([0,b))$ and every $v \in \mathbf{V}$.

Remark 8. If $A_v^1 : L^{\varphi}([0,b),R) \to L^{\varphi}([0,b),\overline{R})$, where $\overline{R} = [-\infty,+\infty]$, then

$$\mathcal{B}_v : M_{Y,\varphi}^o([0,b)) \to M_{Y,\varphi}^o([0,b)).$$

Proof: Let $F \in M_{Y,\varphi}^o([0,b))$, $v \in \mathbf{V}$. We have for $D=[0,b)$

$$\sup_{x\in M(D,Y)\cap F} \| \int_0^b K_v(s-t)x(s)\,ds\|_Y \le \sup_{x\in M(D,Y)\cap F} \{\int_0^b K_v(s-t)\|x(s)\|_Y\,ds\}$$
$$= \int_0^b K_v(s-t)r_F(s)\,ds.$$

On the other hand, for $x(s) = xr_F(s)/\|x\|_Y$ for every $s \in D$, where $x \in Y$ and $x \neq o$, we have

$$\| \int_0^b K_v(s-t)x(s)\,ds\|_Y = \| \frac{x}{\|x\|_Y} \int_0^b K_v(s-t)r_F(s)\,ds\|_Y = \int_0^b K_v(s-t)r_F(s)\,ds.$$

Let $0 < \int_0^b K_v(s-t)r_F(s)\,ds < \infty$ and let $y \in B(o, \int_0^b K_v(s-t)r_F(s)\,ds)$. Let

$$x_t(s) = yr_F(s)/ \int_0^b K_v(s-t)r_F(s)\,ds$$

for every $s \in [0,b)$. We have

$$\int_0^b K_v(s-t)x_t(s)\,ds = y \text{ and } x_t \in M([0,b),Y)\cap F$$

because

$$\|x_t(s)\|_Y = \|yr_F(s)/ \int_0^b K_v(s-t)r_F(s)\,ds\|_Y \le r_F(s) \text{ for every } s\in[0,b).$$

So $\mathcal{B}(F)(t) = B(o, r_{\mathcal{B}(F)}(t))$ for every $t \in [0,b)$, where

$$r_{\mathcal{B}(F)}(t) = \begin{cases} \int_0^b K_v(s-t)r_F(s)\,ds, & \text{if } A_v^1(r_F)(t) < \infty \\ 0, & \text{if } A_v^1(r_F)(t) = \infty \end{cases}$$

for every $t \in [0,b)$. It is easy to see that $r_{\mathcal{B}(F)} \in L^{\varphi}([0,b),R)$.

Let $F \in M_{Y,\varphi}^1([0,b))$ and let $F(s) = B(a_F(s), r_F(s))$ for every $s \in [0,b)$. We introduce the family of operators $T' = (T'_v)_{v\in\mathbf{V}}$ by the formula:

$$T'_v(F)(s) = \begin{cases} B(A_v^2(a_F)(s), A_v^1(r_F)(s)), & \text{if } A_v^1(r_F)(s) < \infty \\ \{A_v^2(a_F)(s)\}, & \text{if } A_v^1(r_F)(s) = \infty \end{cases}$$

for every $s \in [0,b)$ and every $v \in \mathbf{V}$.

Let $F \in \tilde{M}^o_{Y,\varphi}([0,b))$ and $F(s) = \bigcup_{i=1}^n R(o, r^i_F(s), R^i_F(s))$ for every $s \in [0,b)$, where we receive that if there are $D \subset [0,b)$, $D \in \Sigma$, and $m < n$ such that $F(s) = \bigcup_{i=1}^m R(o, \underline{r}^i_F(s), \underline{R}^i_F(s))$, $\underline{R}^i_F(s) < \underline{r}^{i+1}_F(s)$ for $s \in D$, $i = 1, \dots, m-1$ if $m > 1$, then we denote $F(s) = \bigcup_{i=1}^n (o, r^i_F(s), R^i_F(s))$ for every $s \in D$, where $r^i_F(s) = \underline{r}^i_F(s)$, $R^i_F(s) = \underline{R}^i_F(s)$ for $i = 1, \dots, m$, $r^i_F(s) = R^i_F(s) = \underline{R}^i_F(s)$ for $i = m+1, \dots, n$ for every $s \in D$.

We introduce the family of operators $T'' = (T''_v)_{v \in \mathbf{V}}$ by the formula:

$$T''_v(F)(s) = \begin{cases} \bigcup_{i=1}^n R(o, A^1_v(r^i_F)(s), A^1_v(R^i_F)(s)), & \text{if } A^1_v(R^n_F)(s) < \infty \\ \{o\}, & \text{if } A^1_v(R^n_F)(s) = \infty \end{cases}$$

for every $s \in [0,b)$ and every $v \in \mathbf{V}$. □

Remark 9. If $A^1_v : L^\varphi([0,b), R) \to L^\varphi([0,b), \overline{R})$, where $\overline{R} = [-\infty, +\infty]$, then $T'_v : M^1_{Y,\varphi}([0,b)) \to M^1_{Y,\varphi}([0,b))$.

Proof: Let $F \in M^1_{Y,\varphi}([0,b))$, $F(s) = B(a_F(s), r_F(s))$ for every $s \in [0,b)$. It is easy to see that

$$B(A^2_v(a_F)(s), A^1_v(r_F)(s)) = B(A^2_v(a_F)(s), 0) \oplus B(o, A^1_v(r_F)(s))$$

for every $s \in [0,b)$ and $A^2_v : L^\varphi([0,b), Y) \to L^\varphi([0,b), Y)$, so $T'_v(F) \in M^1_{Y,\varphi}([0,b))$. □

Corollary 1. *If the assumptions of Lemma* 2 *and Remarks* 5, 8 *hold, then*

$$T'_v(\mathcal{C}) : M^1_{Y,\varphi}([0,b)) \to M^o_{Y,\varphi}([0,b)).$$

Applying the proofs of Proposition 2 and Theorem 4 in [3], we obtain the following

Theorem 3. *Let φ be a convex, τ-bounded φ-function which fulfils the Δ_2 condition, $\int_0^b \varphi(t,c)\,dt < \infty$ for every $c > 0$ and let $(K_v)_{v \in \mathbf{V}}$ be singular. Then $\varrho(a(A^2_v x - x)) \xrightarrow{\mathcal{V}} 0$ for every $a > 0$ and every $x \in L^\varphi([0,b), Y)$.*

Corollary 2. *If the assumptions of Theorem* 3 *hold, then*

$$T'_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F \text{ for every } F \in M^1_{Y,\varphi}([0,b)).$$

Proof: By the assumptions $T'_v : M^1_{Y,\varphi}([0,b)) \to M^1_{Y,\varphi}([0,b))$. Let $F \in M^1_{Y,\varphi}([0,b))$, $F(s) = B(a_F(s), r_F(s))$ for every $s \in [0,b)$. We have for $a > 0$

$$\int_0^b \varphi(t, a\mathbf{d}(T'_v(F), F)(t))\,dt$$

$$\leq \frac{1}{2} \int_0^b \varphi(t, 2a \mid A^1_v(r_F)(t) - r_F(t) \mid)\,dt$$

$$+\frac{1}{2} \int_0^b \varphi(t, 2a\|A^2_v(a_F)(t) - a_F(t)\|_Y)\,dt \xrightarrow{\mathcal{V}} 0.$$

□

Remark 10. Let $A = \bigcup_{i=1}^{n}[a_i, b_i]$, $B = \bigcup_{i=1}^{n}[c_i, d_i]$, where $[a_i, b_i]$, $[c_i, d_i]$, $i = 1, \dots, n$, are nonempty and compact segments in R, then $\operatorname{dist}(A, B) \le \sum_{i=1}^{n} \operatorname{dist}([a_i, b_i], [c_i, d_i])$.

Corollary 3. *If the assumptions of Theorem 3 hold, then*

$$T''_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F \text{ for every } F \in \tilde{M}^o_{Y,\varphi}([0, b)).$$

Proof: Let $F \in \tilde{M}^o_{Y,\varphi}([0, b))$, $F(s) = \bigcup_{i=1}^{m} R(o, r^i_F(s), R^i_F(s)), a > 0$, $v \in \mathbf{V}$. By the assumptions and by Remark 10 (also, see the proof of Remark 2' and [2, Remark 10]) we have

$$\int_0^b \varphi(t, a\mathbf{d}(T''_v(F), F)(t))\, dt$$
$$\le \frac{1}{2m}\sum_{i=1}^{m}\int_0^b \varphi(t, 2am \mid A^1_v(r^i_F)(t) - r^i_F(t) \mid)\, dt$$
$$+\frac{1}{2m}\sum_{i=1}^{m}\int_0^b \varphi(t, 2am \mid A^1_v(R^i_F)(t) - R^i_F(t) \mid)\, dt \xrightarrow{\mathcal{V}} 0.$$

Let $F \in M^o_{Y,\mathbf{d},\varphi}([0, b))$. Let $v \in \mathbf{V}$ be arbitrary. If there exists $G_v \in M^o_{Y,\mathbf{d},\varphi}([0, b))$ such that $\lim_{n\to\infty}\int_0^b \varphi(t, a\mathbf{d}(T''_v(F_n), G_v)(t))\, dt = 0$ for every $a > 0$ and every sequence $\{F_n\}$ such that $F_n \in \tilde{M}^o_{Y,\varphi}([0, b))$ for every $n \in \mathbf{N}$ and $\lim_{n\to\infty}\int_0^b \varphi(t, a\mathbf{d}(F_n, F)(t))\, dt = 0$ for every $a > 0$, then we define $T_v(F) = G_v$. □

Theorem 4. *Let the assumptions of Theorem 3 hold and there are $K_1, K_2 > 0$ such that $\varrho(a\mathbf{d}(T''_v(F), T''_v(G))) \le K_1\varrho(aK_2\mathbf{d}(F, G))$ for all $F, G \in \tilde{M}^o_{Y,\varphi}([0, b))$, $a > 0$ and every $v \in \mathbf{V}$, then $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in M^o_{Y,\mathbf{d},\varphi}([0, b))$.*

Proof: The proof is analogous to that of Theorem 3' from [2], so we give the sketch of it only. Analogously as in that proof we prove that the family $(T_v)_{v\in\mathbf{V}}$ is $(\mathbf{d}, \mathcal{V}, M^o_{Y,\mathbf{d},\varphi}([0, b)))$-bounded. So we obtain the assertion from Remark 7 and Corollary 3.

Final remarks. The results of [2] can be extended in other ways.

1. Let $x, y \in Y$. By $s(x,y)$ we denote the closed segment joining the points x and y. Let $a \in Y$. Define:

$$Y^a = \{\lambda a : \lambda \in R\},$$

$$Y_\varphi^{1,a} = \{F \in M_Y(I) : F(t) = s(b_F(t), e_F(t)) \text{ for every } t \in I, \text{ where } b_F(\cdot), e_F(\cdot) \in L^\varphi(I, Y^a)\},$$

$$Y_\varphi^{n,a} = \{F \in M_Y(I) : F(t) = \bigcup_{i=1}^{n} s(b_F^i(t), e_F^i(t)) \text{ for every } t \in I, \text{ where } b_F^i(\cdot), e_F^i(\cdot) \in L^\varphi(I, Y^a), i = 1, \dots, n, \|e_F^i(t)\|_Y \le \|b_F^{i+1}(t)\|_Y \text{ for every } t \in I, i = 1, \dots, n-1 \text{ if } n > 1\},$$

$$\tilde{Y}_\varphi^a = \bigcup_{i=1}^{\infty} Y_\varphi^{n,a},$$

$$Y_{\mathbf{d}}^a = \{F \in M_Y(I) : \mathbf{d}(F_n, F) \to 0 \text{ a.e. for some } F_n \in \tilde{Y}_\varphi^a, n \in \mathbf{N}\},$$

$$Y_{\mathbf{d},\varphi}^a = \{F \in Y_{\mathbf{d}}^a : \int_I \varphi(t, \lambda \mathbf{d}(F_n, F)(t)) dt \to 0 \text{ as } n \to \infty \text{ for every } \lambda > 0 \text{ for some } F_n \in \tilde{Y}_\varphi^a, n \in \mathbf{N}\}.$$

The results of [2] will be in force if we replace R by Y, the space $X_{\mathbf{d},\varphi}$ by $Y_{\mathbf{d},\varphi}^a$ and if we introduce the other evident changes.

2. Let $Y = \mathbb{R}^n$. By $\Pi^n(a_i, b_i)$ we denote the Cartesian product of the n closed segments $[a_i, b_i]$, where $a_i, b_i \in \overline{R}$. Define

$$Y_\varphi^{\Pi^n} = \{F \in M_Y(I) : F(t) = \Pi^n(a_i^F(t), b_i^F(t)) \text{ for every } t \in I, a_i^F(\cdot), b_i^F(\cdot) \in L^\varphi(I, Y) \text{ for } i = 1, \dots, n\},$$

$$D(F,G)(t) = \max_{1 \le i \le n} \mathbf{d}([a_i^F, b_i^F], [a_i^G, b_i^G])(t) \text{ for all } F, G \in Y^{\Pi^n},\ t \in I.$$

We easily obtain that the space $\langle Y^{\Pi^n}, \mathbb{D} \rangle$ is a complete space. For all $F \in Y^{\Pi^n}$, $v \in \mathbf{V}$, $t \in [0, b)$ we define:

$$T_v^n(F)(t) = \Pi^n(A_v^1(a_i^F)(t), A_v^1(b_i^F)(t)).$$

We easily obtain the following :

Theorem 5. *If the assumptions of Theorem 3 hold, then*

$$T_v^n(F) \xrightarrow{D,\varphi,\mathcal{V}} F \text{ for every } F \in Y_\varphi^{\Pi^n},\ n \in \mathbf{N}.$$

References

[1] Kasperski A., *Modular approximation in* $\tilde{X}_\varphi$ *by a filtered family of dist-sublinear operators and dist-convex operators*, Mathematica Japonica **38** (1993), 119–125.

[2] ______, *Notes on approximation in the Musielak-Orlicz spaces of multifunctions*, Commentationes Math., in print.

[3] Musielak J., *Modular approximation by a filtered family of linear operators*, Functional Analysis and Approximation, Proc. Conf. Oberwolfach, August 9-16, 1980; Birkhäuser Verlag, Basel, 1981, pp. 99–110.

[4] ______, *Orlicz spaces and Modular spaces*, Lecture Notes in Mathematics Vol. 1034, Springer-Verlag, Berlin, 1983.

[5] Wisła M., *On completeness of Musielak-Orlicz spaces*, Chin. Ann. of Math. **10B (3)** (1989), 292–300.

Institute of Mathematics, Silesian Technical University, Kaszubska 23,
44-100 Gliwice, Poland

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