Rectangular covers of products missing diagonals

Yukinobu Yajima

Abstract. We give a characterization of a paracompact Σ-space to have a G_δ-diagonal in terms of three rectangular covers of $X^2 \setminus \Delta$. Moreover, we show that a local property and a global property of a space X are given by the orthocompactness of $(X \times \beta X) \setminus \Delta$.

Keywords: Σ-space, G_δ-diagonal, σ-closure-preserving, σ-cushioned, rectangular cover, orthocompact, metacompact, Fréchet space

Classification: 54B10, 54D20, 54E18

1. Main theorem

All spaces in this paper are assumed to be regular T_1. The diagonal of a space X is denoted by Δ, that is, $\Delta = \{(x,x) : x \in X\}$.

Let X be a space and $\mathcal{V}$ a collection of subsets of the square X^2. We say that $\mathcal{V}$ is *rectangular* if each member of $\mathcal{V}$ is a subset of the form $U \times W$ in X^2. Note that if $\mathcal{V}$ is a rectangular open cover of $X^2 \setminus \Delta$, then it covers $X^2 \setminus \Delta$ and each member of $\mathcal{V}$ is a subset of the form $U \times W$ such that U and W are disjoint open sets in X.

Gruenhage and Pelant [4] proved that a paracompact Σ-space X has a G_δ-diagonal (i.e. is a σ-space), if $X^2 \setminus \Delta$ is paracompact. Subsequently, Kombarov [7] proved that a paracompact Σ-space X has a G_δ-diagonal if and only if there is a locally finite rectangular open cover of $X^2 \setminus \Delta$.

Our main theorem is an extension of these results in terms of three rectangular covers of $X^2 \setminus \Delta$.

Theorem 1. *The following are equivalent for a paracompact Σ-space X.*

- (a) *X has a G_δ-diagonal.*
- (b) *There is a σ-locally finite rectangular open cover of $X^2 \setminus \Delta$.*
- (c) *There is a σ-closure-preserving rectangular open cover of $X^2 \setminus \Delta$.*
- (d) *There is a rectangular open cover of $X^2 \setminus \Delta$ which has a σ-cushioned open refinement.*

The author announced Theorem 1 except (c) $\Rightarrow$ (a) in [10], and asked whether (c) implies (a) in the conference. Answering this, we give a complete proof of Theorem 1 in the next section.

2. Proof of Theorem 1

The main idea of the proof of Theorem 1 is based on Gruenhage-Pelant's. In fact, it will be proceeded along the line of that of [4, Theorem 4]. Here, we have to do two kinds of parallel arguments.

Recall that a collection of $\mathcal{V}$ of subsets of a space X is *closure-preserving* if $\overline{\bigcup\{V : V \in \mathcal{V}'\}} = \bigcup\{\overline{V} : V \in \mathcal{V}'\}$ for each $\mathcal{V}' \subset \mathcal{V}$. We say that $\mathcal{V}$ is *σ-closure-preserving* if it can be written as $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ such that each $\mathcal{V}_n$ is closure-preserving.

Lemma 1. *Let X be a space with $p \in X$. Let $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ be a σ-closure-preserving rectangular open cover of $X^2 \setminus \Delta$. If there is a countable subset M of $X \setminus \{p\}$ such that $p \in \overline{M}$, then p is a G_δ-point.*

Proof: Let $M = \{x_n : n \in \omega\}$. Let $F_n = \bigcup\{\overline{V} : V \in \mathcal{V}_n,\ j \leq n \text{ with } (p, x_n) \notin \overline{V}\}$ for each $n \in \omega$. Then each F_n is a closed subset in $X^2 \setminus \Delta$ such that $(p, x_n) \notin F_n$. For each $n \in \omega$, take a basic open neighborhood $G_n \times H_n$ of (p, x_n) in $X^2 \setminus \Delta$, disjoint from F_n. We show $\bigcap_{n\in\omega} G_n = \{p\}$. Assume that there is some $y \in \bigcap_{n\in\omega} G_n$ with $y \neq p$. Take some $V = U \times W \in \mathcal{V}$ such that $(y, p) \in V$. Choose $m \in \omega$ with $V \in \mathcal{V}_m$. By $p \notin \overline{U}$, we have $(p, x_n) \notin \overline{V}$. Hence it follows that $\overline{V} \subset F_n$ for each $n \geq m$. Choose some $k \geq m$ with $x_k \in W$. Then it follows that $(y, x_k) \in U \times W = V \subset \overline{V} \subset F_k$. On the other hand, by $(y, x_k) \in G_k \times H_k$, we have $(y, x_k) \notin F_k$. This is a contradiction. $\square$

Let $\mathcal{V}$ and $\mathcal{O}$ be two collections of subsets of a space X. Recall that $\mathcal{V}$ is *cushioned in $\mathcal{O}$* if for each $V \in \mathcal{V}$, one can assign an $O(V) \in \mathcal{O}$ such that for each $\mathcal{V}' \subset \mathcal{V}$, $\overline{\bigcup\{V : V \in \mathcal{V}'\}} \subset \bigcup\{O(V) : V \in \mathcal{V}'\}$. Such an assignment $V \mapsto O(V)$, $V \in \mathcal{V}$, is called a *cushioned assignment* from $\mathcal{V}$ into $\mathcal{O}$. We say that $\mathcal{V}$ is *σ-cushioned in $\mathcal{O}$* if it can be written as $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ such that each $\mathcal{V}_n$ is cushioned in $\mathcal{O}$.

Lemma 2. *Let X be a space with $p \in X$. Let $\mathcal{O}$ be a rectangular open cover of $X^2 \setminus \Delta$. Let $\mathcal{V}$ be a collection of open sets in $X^2 \setminus \Delta$ which is cushioned in $\mathcal{O}$. If there is a countable subset M of $X \setminus \{p\}$ such that $p \in \overline{M}$ and $M \times \{p\} \subset \bigcup \mathcal{V}$, then p is a G_δ-point.*

Proof: Let $V \mapsto O(V)$ be a cushioned assignment from $\mathcal{V}$ into $\mathcal{O}$, and let $M = \{x_n : n \in \omega\}$. For each $n \in \omega$, take $V_n \in \mathcal{V}$ with $(x_n, p) \in V_n$, and let $W_n = \{x \in X : (x_n, x) \in V_n\}$. It suffices to show $\bigcap_{n\in\omega} W_n = \{p\}$. Assume that there is some $y \in \bigcap_{n\in\omega} W_n$ with $y \neq p$. Let $O(V_n) = P_n \times Q_n$ for each $n \in \omega$. By $(x_n, p) \in V_n \subset O(V_n)$, we have $p \notin \overline{P_n}$ for each $n \in \omega$. So it follows that $(p, y) \notin \bigcup_{n\in\omega}(P_n \times Q_n) = \bigcup_{n\in\omega} O(V_n) \supset \overline{\bigcup_{n\in\omega} V_n}$. There is an open neighborhood G of p such that $(G \times \{y\}) \cap (\bigcup_{n\in\omega} V_n) = \emptyset$. By $(x_n, y) \in V_n$, each x_n is not in G. Hence we have $p \notin \overline{M}$, which is a contradiction. $\square$

A space X is called a *Σ-space* if there are a closed cover $\mathcal{C}$ of X by countably compact sets, and a σ-discrete closed cover $\mathcal{F}$ of X such that whenever $C \in \mathcal{C}$ and U is open in X with $C \subset U$, then $C \subset F \subset U$ for some $F \in \mathcal{F}$.

The class of paracompact Σ-spaces is a broad one which is countably productive (see [3], [8]).

Lemma 3. *Let X be a Σ-space. If there is a σ-closure-preserving rectangular open cover $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ of $X^2 \setminus \Delta$, then each point of X is G_δ.*

Proof: Let $\mathcal{C}$ and $\mathcal{F}$ be two closed covers of X, described as above. Assume that some $p \in X$ is not a G_δ-point. As stated in the proof of [4, Lemma 3], there is a closed G_δ-set Y in X containing p such that $y \in Y$ and $y \in C \in \mathcal{C}$ implies $p \in C$. Then note that p is not G_δ in Y. Let $V = U_V \times W_V$ for each $V \in \mathcal{V}$. Let $\mathcal{U}_n = \{U_V : V \in \mathcal{V}_n \text{ with } p \in W_V\}$ for each $n \in \omega$. Note that each $\mathcal{U}_n$ is closure-preserving in $X \setminus \{p\}$. Let $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$. Then $\mathcal{U}$ is a σ-closure-preserving open cover of $X \setminus \{p\}$.

Now, we construct $\{U_\alpha, G_\alpha, z_\alpha : \alpha \in \omega_1\}$, satisfying the following conditions; for each $\alpha \in \omega_1$,

(i) $U_\alpha \in \mathcal{U}$ and G_α is an open set in Y,

(ii) $\overline{U}_\alpha \cap Y \subset G_\alpha \subset \overline{G}_\alpha \subset Y \setminus \{p\}$,

(iii) $z_\alpha \in (U_\alpha \cap (Y \setminus \{p\})) \setminus \bigcup_{\beta<\alpha} G_\beta$.

In fact, for $\alpha \in \omega_1$, assume that $\{U_\beta, G_\beta, z_\beta : \beta < \alpha\}$ satisfies the above conditions. Since $Y \setminus \{p\}$ is not $F_\sigma$in Y, $\{\overline{G}_\beta : \beta < \alpha\}$ does not cover $Y \setminus \{p\}$. However, as $\mathcal{U}$ covers $X \setminus \{p\}$, we can choose the desired z_α and U_α. By the choice of $\mathcal{U}_n$, note that $p \notin \overline{U}_\alpha$. Since X is regular, we can choose the desired G_α.

Here we may assume without loss of generality that $\{U_\alpha : \alpha \in \omega_1\} \subset \mathcal{U}_m$ for some $m \in \omega$. Let $Z = \{z_\alpha : \alpha \in \omega_1\}$. Then Z is uncountable. Moreover, Z is closed discrete in $Y \setminus \{p\}$. For, pick any $x \in \overline{Z} \setminus \{p\}$. Since $\mathcal{U}_m$ is closure-preserving, $\overline{Z} \setminus \{p\} \subset \bigcup_{\alpha\in\omega_1} \overline{U}_\alpha$. Let $\alpha_0 = \min\{\alpha \in \omega_1 : x \in \overline{U}_\alpha\}$. Let $N = G_{\alpha_0} \setminus \bigcup_{\beta<\alpha_0} \overline{U}_\beta$. Then N is an open neighborhood of x in Y such that $N \cap Z \subset \{z_{\alpha_0}\}$.

It follows from Lemma 1 that $p \notin \overline{M}$ for each countable subset M of Z. The remaining argument is the same as in the proof of [4, Lemma 3]. □

Lemma 4. *Let X be a Σ-space. If there is a rectangular open cover $\mathcal{O}$ of $X^2 \setminus \Delta$ which has a σ-cushioned open refinement $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$, then each point of X is G_δ.*

Proof: Let $\mathcal{C}$, $\mathcal{F}$, p and Y be the same as in the above proof. Let $V \mapsto O(V)$, $V \in \mathcal{V}_n$, be a cushioned assignment of $\mathcal{V}_n$ into $\mathcal{O}$. Let $U_V = \{x \in X : (x, p) \in V\}$ and $O(V) = P_V \times Q_V$ for each $V \in \mathcal{V}$. Moreover, let $\mathcal{U}_n = \{U_V : V \in \mathcal{V}_n\}$ and $\mathcal{P}_n = \{P_V : V \in \mathcal{V}_n \text{ with } p \in Q_V\}$ for each $n \in \omega$. Let $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ and $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$. Then $\mathcal{U}$ is an open cover of $X \setminus \{p\}$ and each $\mathcal{U}_n$ is cushioned in $\mathcal{P}_n$ with the cushioned assignment $U_V \mapsto P_V$ in $X \setminus \{p\}$. Since $p \notin \overline{P}$ for each $P \in \mathcal{P}$ and $Y \setminus \{p\}$ is not F_σ in Y, $\{P \cap Y : P \in \mathcal{P}\}$ has no countable subcover of $Y \setminus \{p\}$. So we can inductively choose $\{U_\alpha, P_\alpha, z_\alpha : \alpha \in \omega_1\}$, satisfying for each $\alpha \in \omega_1$,

(iv) $U_\alpha \in \mathcal{U}$ and $P_\alpha \in \mathcal{P}$,

(v) $z_\alpha \in (U_\alpha \cap (Y \setminus \{p\})) \setminus \bigcup_{\beta<\alpha} P_\beta$.

We may assume that $\{U_\alpha : \alpha \in \omega_1\} \subset \mathcal{U}_m$ and $\{P_\alpha : \alpha \in \omega_1\} \subset \mathcal{P}_m$ for some $m \in \omega$. Then $U_\alpha \mapsto P_\alpha$, $\alpha \in \omega_1$, is a cushioned assignment. Let $Z = \{z_\alpha : \alpha \in \omega_1\}$. Similarly, Z is closed discrete in $Y \setminus \{p\}$. Here, using Lemma 2 instead of Lemma 1, the remaining argument is the same as above. □

A space X is called a *β-space* if there is a function $g : \omega \times X \to \tau(X)$, where $\tau(X)$ denotes the topology of X, satisfying for each $x \in X$,

(i) $x \in \bigcap_{n\in\omega} g(n, x)$,
(ii) if $x \in g(n, x_n)$ for each $n \in \omega$, then $\{x_n : n \in \omega\}$ has a cluster point in X.

Since Σ-spaces and semi-stratifiable spaces are β-spaces (see [3, Theorem 7.8 (i)]), the class of β-spaces is fairly broad.

The definition of W_δ-diagonal in terms of the notion of a sieve is seen in [3], [4]. We do not restate it here. It is shown in [2] (or [3, Theorem 6.6]) that a submetacompact space with a W_δ-diagonal has a G_δ-diagonal.

Lemma 5. *Let X be a β-space such that each point of X is G_δ. If there is a σ-closure-preserving rectangular open cover $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ of $X^2 \setminus \Delta$, then X has a W_δ-diagonal.*

Proof: Let $g : \omega \times X \to \tau(X)$ be a function, described as above. Let $h : \omega \times X \to \tau(X)$ be a function such that $\bigcap_{n\in\omega} h(n, x) = \{x\}$ for each $x \in X$. As in the proof of [4, Theorem 4], we can construct a sieve $(G, X^{<\omega})$, satisfying the following: If $s = \langle x_0, \dots, x_{n-1}\rangle \in X^{<\omega}$ and $x \in G(s)$, then $G(s^\frown\langle x\rangle)$ is an open neighborhood of x such that

(i) $\overline{G(s^\frown\langle x\rangle)} \subset G(s) \cap g(n, x) \cap h(n, x)$,
(ii) if $i < n$ and $x_i \neq x$, then

$$(\{x_i\} \times G(s^\frown\langle x\rangle)) \cap (\bigcup\{\overline{V} : V \in \mathcal{V}_j, j \le n \text{ with } (x_i, x) \notin \overline{V}\}) = \emptyset\,.$$

Assume that $\bigcap_{n\in\omega} G(s \restriction n)$ contains two distinct points for some $s = \langle x_0, x_1, \dots\rangle \in X^\omega$. Then, by (i), no point of X is repeated infinitely many times in the sequence s. By the choice of g and (i), $\{x_n : n \in \omega\}$ has a cluster point y. Then we have $y \in \bigcap_{n\in\omega} \overline{G(s \restriction n)} = \bigcap_{n\in\omega} G(s \restriction n)$. There is some $z \in \bigcap_{n\in\omega} G(s \restriction n)$ with $y \neq z$. Choose an $n_0 \in \omega$ and a $V_0 = U \times W \in \mathcal{V}_{n_0}$ with $(y, z) \in V_0$. Find some $k, m \in \omega$ such that $m > k > n_0$, $x_k \neq x_m$ and $\{x_k, x_m\} \subset U$. By $x_m \notin \overline{W}$, note $(x_k, x_m) \notin \overline{V_0}$. By (ii), we have $(\{x_k\} \times G(s \restriction m + 1)) \cap \overline{V_0} = \emptyset$. On the other hand, we have $(x_k, z) \in (\{x_k\} \times G(s \restriction m + 1)) \cap \overline{V_0}$. This is a contradiction. □

Lemma 6. *Let X be a β-space such that each point of X is G_δ. If there is a rectangular open cover $\mathcal{O}$ of $X^2 \setminus \Delta$ which has a σ-cushioned open refinement $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$, then X has a W_δ-diagonal.*

Proof: Let $V \mapsto O(V)$ be a cushioned assignment from $\mathcal{V}_n$ into $\mathcal{O}$ for each $n \in \omega$. Let g and h be the same functions as above. Moreover, we can also construct a similar sieve $(G, X^{<\omega})$ as above, where we only replace the condition (ii) with the following;

(ii$'$) if $i < n$ and $x_i \neq x$, then

$$(\{x_i\} \times G(s^\frown\langle x\rangle)) \cap (\bigcup\{V \in \mathcal{V}_j, j \leq n, (x_i, x) \notin O(V)\}) = \emptyset\,.$$

Take s, y and z as above. Choose an $n_0 \in \omega$ and a $V_0 \in \mathcal{V}_{n_0}$ with $(y, z) \in V_0$. Take an open neighborhood U of y such that $U \times \{z\} \subset V_0$. Find some $k, m \in \omega$ with $m > k > n_0$, $x_k \neq x_m$ and $\{x_k, x_m\} \subset U$. Let $O(V_0) = P \times Q$. Since $x_m \in U \subset P$, it follows that $(x_k, x_m) \notin O(V_0)$. By (ii$'$), we have $(\{x_k\} \times G(s \restriction m+1)) \cap V_0 = \emptyset$. On the other hand, we have

$$(x_k, z) \in (\{x_k\} \times G(s \restriction m+1)) \cap (U \times \{z\}) \subset (\{x_k\} \times G(s \restriction m+1)) \cap V_0\,.$$

This is a contradiction. □

We say that an open cover $\mathcal{O}$ of $X^2 \setminus \Delta$ is *rectangular cozero* if each member of $\mathcal{O}$ is a subset of the form $P \times Q$ such that P and Q are disjoint cozero sets in X.

Since each open F_σ-set in a normal space is exactly a cozero set, so is each open set in a metric space. So, Kombarov [7] actually showed the following.

Lemma 7. *If a paracompact space X has a G_δ-diagonal, then there is a locally finite rectangular cozero cover of $X^2 \setminus \Delta$.*

Now, we complete the proof of our main theorem.

Proof of Theorem 1: (a) $\Rightarrow$ (b): This follows from Lemma 7 (or [7, Theorem 1]).

(b) $\Rightarrow$ (c): Obvious.

(a) $\Rightarrow$ (d): Since a σ-locally finite rectangular cozero cover of $X^2 \setminus \Delta$ has a σ-cushioned (rectangular) open refinement, this also follows from Lemma 7.

(c) $\Rightarrow$ (a): Remember that each Σ-space is a β-space, and that a submetacompact space has a G_δ-diagonal iff it has a W_δ-diagonal. So this follows from Lemmas 3 and 5.

(d) $\Rightarrow$ (a): Similarly, this follows from Lemmas 4 and 6. □

3. Orthocompactness of $(X \times \beta X) \setminus \Delta$

Arhangel'skiĭ and Kombarov [1] proved that a compact space X is first countable if $X^2 \setminus \Delta$ is normal. First, we consider what local property of a compact space X can be obtained if the normality of $X^2 \setminus \Delta$ is replaced by the orthocompactness of it. For this, we also use some rectangular open covers.

Recall that an open cover $\mathcal{V}$ of a space X is *interior-preserving* if $\bigcap \mathcal{V}'$ is open in X for each $\mathcal{V}' \subset \mathcal{V}$.

A space X is called a *Fréchet space* if for each $p \in X$ and each subset M of X with $p \in \overline{M}$, there is a sequence $\{x_n\}$ of points in M which converges to p.

Note that point-finite open covers of a space are interior-preserving, and that first countable spaces and Lašnev spaces are Fréchet.

For a collection $\mathcal{V}$ of open sets in a product $X \times C$ and an $(x, y) \in X \times C$, let $\bigcap \mathcal{V}(x, y) = \bigcap\{V \in \mathcal{V} : (x, y) \in V\}$.

Theorem 2. *Let C be a countably compact space and X a subspace of C. If there is a rectangular open cover of $(X \times C) \setminus \Delta$ which has an interior-preserving open refinement, then X is a Fréchet space.*

Proof: Let M be a subset of X with $p \in \overline{M} \setminus M$. Let $\mathcal{O}$ be a rectangular open cover of $(X \times C) \setminus \Delta$ and $\mathcal{V}$ an interior-preserving open refinement of $\mathcal{O}$. Since p is not isolated in X and each $\bigcap \mathcal{V}(p, x)$ is an open neighborhood of (p, x) in $X \times C$, we can inductively choose a sequence $\{x_n : n \in \omega\}$ of distinct points in M such that $(x_n, x_i) \in \bigcap \mathcal{V}(p, x_i)$ for each $i < n$ and each $n \in \omega$. We show that $\{x_n : n \in \omega\}$ converges to p. Assume the contrary. This is an open neighborhood U of p in X such that $x_n \notin U$ for infinitely many n's. There is a cluster point y of $\{x_n \in X \setminus U : n \in \omega\}$ in C. By $y \neq p$, we can find a $V \in \mathcal{V}$ and an $O = P \times Q \in \mathcal{O}$ such that $(p, y) \in V \subset O$. Take an open neighborhood W of y in C such that $\{p\} \times W \subset \bigcap \mathcal{V}(p, y)$. Moreover, take some $k, m \in \omega$ such that $k < m$ and $\{x_k, x_m\} \subset W$. Since $(p, x_k) \in \bigcap \mathcal{V}(p, y)$, it follows that $\bigcap \mathcal{V}(p, x_k) \subset \bigcap \mathcal{V}(p, y)$. Hence we have

$$(x_m, x_k) \in \bigcap \mathcal{V}(p, x_k) \subset \bigcap \mathcal{V}(p, y) \subset V \subset O = P \times Q\,.$$

On the other hand, we have

$$(p, x_m) \in \{p\} \times W \subset \bigcap \mathcal{V}(p, y) \subset V \subset O = P \times Q\,.$$

Thus we obtain $x_m \in P \cap Q$. This is a contradiction. □

The author first showed in Theorem 2 that X has countable tightness. Subsequently, N. Kemoto kindly pointed out that X is a Fréchet space.

We say that a space X is *orthocompact* if every open cover of X has an interior-preserving open refinement.

As an analogue of [1, Theorem 10], we immediately have

Corollary 1. *Let X be a countably compact space. If $X^2 \setminus \Delta$ is orthocompact, then X is a Fréchet space.*

For a Tychonoff space X, we denote by βX the Stone-Čech compactification of X. Junnila [5] proved that the orthocompactness of $X \times \beta X$ gives the metacompactness of X. Finally, we show that the orthocompactness of $(X \times \beta X) \setminus \Delta$ gives not only the local property of X but also the global property of X.

Theorem 3. *Let X be a Tychonoff space and γX a compactification of X. If $(X \times \gamma X) \setminus \Delta$ is orthocompact, then X is metacompact.*

Proof: The proof is obtained by modifying that of [9, Theorem 2.2]. Let $\mathcal{U}$, $\mathcal{U}^*$, $\mathcal{V}$ and $\mathcal{G}$ be the same ones as in the proof of it. There is an interior-preserving open refinement $\mathcal{H}$ of $\mathcal{G}|(X \times \gamma X) \setminus \Delta$. For each $x \in X$, fix a $V_x \in \mathcal{V}$ with $x \in V_x$. For each $(x, x') \in (X \times \gamma X) \setminus \Delta$, we take a basic open neighborhood $P_{x,x'} \times Q_{x,x'}$ of (x, x') which is contained in some member of $\mathcal{H}$. Pick $x \in X$. Since $\gamma X \setminus V_x$ is compact, there is a finite subset $F(x)$ of $\gamma X \setminus V_x$ such that $\gamma X \setminus V_x \subset \bigcup_{z\in F(x)} Q_{x,z}$. Let $W_x = (\bigcap_{z\in F(x)} P_{x,z}) \cap V_x$. Here, we set $\mathcal{W} = \{W_x : x \in X\}$. It suffices from [6, Theorem 3.6] to show that there is a finite subcollection $\mathcal{U}_x$ of $\mathcal{U}$ such that $\mathrm{St}(x, \mathcal{W}) \subset \bigcup \mathcal{U}_x$ and $x \in \bigcap \mathcal{U}_x$ for each $x \in X$. For this, it also suffices to show that $\mathrm{Cl}_{\gamma X} \mathrm{St}(x, \mathcal{W}) \subset \mathrm{St}(x, \mathcal{U}^*)$ for each $x \in X$. Assuming the contrary, we pick some $x \in X$ and some $q \in \mathrm{Cl}_{\gamma X} \mathrm{St}(x, \mathcal{W}) \setminus \mathrm{St}(x, \mathcal{U}^*)$. By $x \neq q$, $\bigcap \mathcal{H}(x, q)$ is an open neighborhood of (x, q). Take a basic open neighborhood $S \times T$ of (x, q) contained in $\bigcap \mathcal{H}(x, q)$. Pick $p \in T \cap \mathrm{St}(x, \mathcal{W})$, and pick $y \in X$ with $x \in W_y$ and $p \in W_y$. Since $x \in W_y \subset V_y \subset U^*_{V_y} \in \mathcal{U}^*$, it follows that

$$q \in \gamma X \setminus \mathrm{St}(x, \mathcal{U}^*) \subset \gamma X \setminus V_y \subset \bigcup_{z\in F(y)} Q_{y,z} .$$

Find $z \in F(y)$ with $q \in Q_{y,z}$. By the same argument as in the proof of [9, Theorem 2.2], we obtain that $\{(x, q), (x, p), (p, q)\} \subset H_0$ for some $H_0 \in \mathcal{H}$, and so that $\{(x, p), (p, q)\} \subset (V_0 \cap X) \times (\gamma X \setminus \mathrm{Cl}_{\gamma X} V_0)$ for some $V_0 \in \mathcal{V}$. This is a contradiction. □

By Theorems 2 and 3, we obtain

Corollary 2. *Let X be a Tychonoff space. If $(X \times \beta X) \setminus \Delta$ is orthocompact, then X is metacompact and Fréchet.*

References

[1] Arhangel'skiĭ A.V., Kombarov A.P., *On ∇-normal spaces*, Topology and Appl. **35** (1990), 121–126.

[2] Chaber J., Čoban M., Nagami N., *On monotonic generalizations of Moore spaces, Čech-complete spaces, and p-spaces*, Fund. Math. **83** (1974), 107–119.

[3] Gruenhage G., *Generalized metric spaces*, Handbook of Set-theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 423–501.

[4] Gruenhage G., Pelant J., *Analytic spaces and paracompactness of $X^2 \setminus \Delta$*, Topology and Appl. **28** (1988), 11–15.

[5] Junnila H.J.K., *Metacompactness, paracompactness and interior-preserving open covers*, Trans. Amer. Math. Soc. **249** (1979), 373–385.

[6] ———, *On submetacompactness*, Topology Proc. **3** (1978), 375–405.

[7] Kombarov A.P., *On rectangular covers of $X^2 \setminus \Delta$*, Comment. Math. Univ. Carolinae **30** (1989), 81–83.

[8] Nagami K., *Σ-spaces*, Fund. Math. **65** (1969), 169–192.

[9] Yajima Y., *A characterization of submetacompactness in terms of products*, Proc. Amer. Math. Soc. **112** (1991), 291–296.

[10] ———, *Subspaces of squares; $X^2 \setminus \Delta$ and others*, Abstracts of Short Conference of Uniform Mathematics and its Applications, Bern, 1991.

Department of Mathematics, Kanagawa University, Yokohama 221, Japan

(*Received* May 20, 1993)