Cardinal invariants and compactifications

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Abstract. We prove that every compact space X is a Čech-Stone compactification of a normal subspace of cardinality at most $d(X)^{t(X)}$, and some facts about cardinal invariants of compact spaces.

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The notions of a compactness and of a compactification are very closed. Really, any compact space is a compactification of any of its dense subsets. But a useful information can be obtained from this fact only in the case when a compact space is a compactification of some certain type.

The classic type is the Čech-Stone compactification. When a compact space is a Čech-Stone compactification of some of its subsets? Theorem 1.1 says that any compact space X is a Čech-Stone compactification of some normal subspace $B \subseteq X$ such that $|B| \leq d(X)^{t(X)}$, where $d(X)$ is a density of X, $t(X)$ is its tightness. The case, when $|X| \leq d(X)^{t(X)}$ is trivial, in this case $B = X$. In the opposite case the theorem says that "extra" points from $X \setminus B$ are constructed by a standard way, as points of Čech-Stone compactification of a "not large" normal subspace. For example, by this theorem, we can say that Fedorchuk's compact space, that is the hereditarily separable, hereditarily normal compact space of cardinality $2^{\mathfrak{c}}$, is a Čech-Stone compactification of a subspace of a cardinality $\mathfrak{c}$. We prove some new facts about cardinal invariants.

The definitions and notations used here are standard, one can find then in [2], for example.

We use the notation of a sequential extension of a set A, that is a set B, $A \subseteq B \subseteq [A]$, $|B| \leq |A|^\omega$ such that if a countable set $B' \subseteq B$ has a limit point in X, then there is a limit point of B' in B. One can construct a sequential extension by induction.

Theorem 1.1. *Let X be a compact space, A be a dense subset of X. Then $X = \beta B$, where $A \subseteq B$, $|B| \leq |A|^{t(X)}$, B is a normal and countably compact subspace.*

Proof: The case $|X| = |A|^{t(X)}$ is trivial. Let $|X| > |A|^{t(X)}$. We construct by induction the family $\{B_\alpha : \alpha < \omega_{\tau^+}\}$, where $t(X) = \tau$ such that:

(1) $B_0 = A$;
(2) $B_\beta \subseteq B_\alpha$ for $\beta \leq \alpha$;
(3) a sequential extension of B_α is in $B_{\alpha+1}$;
(4) $|B_\alpha| \leq |A|^\tau$ for $\alpha < \omega_{\tau^+}$;
(5) $\bigcup\{B_\alpha : \alpha < \omega_{\tau^+}\} = B$, where B is a normal, countably compact subspace and $X = \beta B$.

Let $\xi(A)$ be a choice function, defined on the set of all nonempty subsets of X. Let $B_0 = A$. Let $\{B_\beta : \beta < \alpha\}$ with conditions (1)–(4) be constructed. Let B'_α be a sequential extension of the set $\bigcup\{B_\beta : \beta < \alpha\}$ and

$$B_\alpha = B'_\alpha \cup \{\xi([T]_X \cap [T']_X) : T, T' \in \exp_\tau B'_\alpha\}.$$

Then $|B_\alpha| \leq |A|^\tau$ and (1)–(4) hold for $\{B_\beta : \beta \leq \alpha\}$. Note that B is a countably compact space. Let us prove that $X = \beta B$. Let F_1, F_2 be disjoint closed subsets of B. Let $([F_1]_X \cap [F_2]_X) \setminus B \neq \emptyset$ and $x \in ([F_1]_X \cap [F_2]_X) \setminus B$. Since $t(X) = \tau$, there are $F'_1 \subseteq F_1$ and $F'_2 \subseteq F_2$ such that $F'_1, F'_2 \subseteq B_\alpha$ and $|F'_1| \leq \tau$, $|F'_2| \leq \tau$, and $x \in [F'_1]_X \cap [F'_2]_X$. There is $\alpha < \omega_{\tau^+}$ such that $F'_1, F'_2 \subseteq B_\alpha$. Since F'_1, F'_2 are disjoint, closed subsets of B, then $[F'_1]_X \cap [F'_2]_X \subseteq ([F_1]_X \cap [F_2]_X) \setminus B$. Then $\xi([F'_1]_X \cap [F'_2]_X) \in X \setminus B$, a contradiction. So $X = \beta B$. It follows from the above proof that B is a normal space. The theorem is proved. $\square$

Recall that a space X is weakly normal if in every closed, countable, discrete set $A \subseteq X$ there is a countable $A' \subseteq A$ C^*-embedded in X [3].

We say that X is an h-weakly normal space if X is hereditarily weakly normal.

Definition 1.2. A space X is called d-normal if in every closed, countable, discrete set $A \subseteq X$ there is a countable $A' \subseteq A$ with discrete family of neighborhoods. It means that for every point $x \in A'$ there is a neighborhood Ox such that $\{Ox : x \in A'\}$ is a discrete family.

We say that a space X is hd-normal if X is a hereditarily d-normal.

It is clear that a hd-normal (d-normal) space is h-weakly (weakly) normal, and a hereditarily normal space is hd-normal as well as it is compact first countable space or a normal first countable space.

On the other hand, the space $N^* = \beta N \setminus N$, the remainder of the Čech-Stone compactification of a countable discrete space, is an h-weakly normal, but not an hd-normal space. Really, the h-weakly normality of N^* follows from the fact that $[D]_{N^*}$ is homeomorphic to $\beta N = \beta D$ for every countable discrete set $D \subseteq N^*$. But for a discrete set D the space $X = N^* \setminus ([D]_{N^*} \setminus D)$ is not d-normal.

Theorem 1.3. *Let X be an hd-normal compact space, or an h-weakly normal space with countable tightness, and $A \subseteq X$ be a dense subset of X. Then $X = \beta B$ where $A \subseteq B$, $|B| \leq |A|^{t(X)}$, B is a normal countably compact space such that every compact space $K \subseteq X \setminus B$ is finite.*

Proof: Again, we suppose that $|X| > |A|^{t(X)}$. Let the set B be as in Theorem 1.1. We prove that every compact space $K \subseteq X \setminus B$ is finite. Let $K \subseteq X \setminus B$

be an infinite compact set. There is a countable discrete (as a subspace) subset $D \subseteq K$. We consider a set $B \cup D$. By h-weakly normality (moreover hd-normality) of X, there is a countable set $D' \subseteq D$ C^*-embedded in $B \cup D$. Then $[D']_X = \beta D'$, so $[D']_X$ is a Čech-Stone compactification of the countable discrete set. But $\beta D' = \beta N$ is not an hd-normal and the tightness of $\beta D'$ is not countable. This contradiction proves the theorem. □

Theorems 1.1 and 1.3 were announced by the author in [4].

Lemma 1.4. *Let X be an h-weakly normal compact space with countable tightness, $X = \beta B$ for some $B \subseteq X$. The $\Phi \setminus F$ is a discrete (as a subspace) set for closed $F, \Phi \subseteq X$ such that $F \subseteq \Phi$ and $F \cap B = \Phi \cap B$.*

Proof: Let $x \in \Phi \setminus F$. There is a neighborhood Ox of x such that $[Ox]_X \cap F = \emptyset$. Therefore, $[Ox]_X \cap \Phi = [Ox]_X \cap (\Phi \setminus F) \subseteq X \setminus B$. By Theorem 1.3, the set $[Ox]_X \cap \Phi$ is finite, and therefore $\Phi \setminus F$ is a discrete set. The lemma is proved. □

Recall that a point $x \in X$ is called a b-point if $x = F \cap \Phi$ where F and Φ are closed sets in X, and x is a limit point for F and Φ [5].

$\varkappa$-points (limits of sequences of points of X) are b-points as well as points of non-normality (points x of a normal space X such that $X \setminus \{x\}$ is non-normal).

Theorem 1.5. *Let X be an h-weakly normal compact space with countable tightness. Then $|\{x : x \text{ is a } b\text{-point in } X\}| \leq d(X)^\omega$.*

Proof: By Theorem 1.3, $X = \beta B$ for a normal B such that $|B| \leq d(X)^\omega$ and every compact subset of $X \setminus B$ is finite. We prove that none of the points of $X \setminus B$ is a b-point. Indeed, let $x \in X$ be a b-point, that is, $x = F \cap \Phi$ where F, Φ are closed in X, and x is a limit point for F and Φ. Let $F' = F \cap B$, $\Phi' = \Phi \cap B$. Then $x \in [F']_X \cap [\Phi']_X$. Really, by Lemma 1.4, the sets $F \setminus [F']_X$ and $\Phi \setminus [\Phi']_X$ are discrete and therefore $x \in [F']_X$, $x \in [\Phi']_X$. But X is a Čech-Stone compactification of the normal space B. This contradiction proves the theorem. □

Corollary 1.6. *Let X be a weakly normal compact space with countable tightness and let all points of X be b-points. Then $|X| \leq d(X)^\omega$.*

The above results have some connections with the results from [6].

Recall that $x \in X$ is an hb-point if x is a b-point in every closed subspace $X' \subseteq X$, if x is not isolated in X' [5].

By Arhangelskii's theorem [7] and Theorem 1.5 we have

Theorem 1.7. *Let X be an h-weakly normal compact space with countable tightness, $\chi(X) \leq 2^\omega$ and every non-isolated point in X is an hb-point. Then $|X| \leq 2^\omega$.*

Proposition 1.8. *Let X be a countably compact hd-normal space, $A \subseteq X$ be a dense subset. Then there is $B \subseteq X$ such that $A \subseteq B$, $|B| \le |A|^\omega$, B is countably compact such that every subset $F \subseteq X \setminus B$ closed in X is finite.*

Proof: It is clear that B is a sequential extension of A. The only thing we have to explain is the last part. Let $F \subseteq X \setminus B$ be an infinite closed subset of X. There is a countable discrete (as a subspace) set $D \in F$. By an hd-normality of X there is a countable set $D' = \{x_i : i \in \omega\}$, $D' \subseteq D$ with a discrete in $B \cup D$ family of neighborhoods $\{Ox_i : i \in \omega\}$. But $Ox_i \cap B \ne \emptyset$ for every $i \in \omega$, so we have the discrete infinite family in the compact space B. This contradiction proves the proposition. □

By the same way as Lemma 1.4 we can prove

Lemma 1.9. *Let X be an hd-normal space, $B \subseteq X$ be a dense, countably compact subspace, $F, \Phi \subseteq X$ such that $F \subseteq \Phi$, $F \cap B = \Phi \cap B$. Then $\Phi \setminus F$ is a discrete (as a subspace) set, moreover, $\Phi \setminus F$ is a free sequence in $X \setminus F$.*

Lemma 1.10. *Let X be an hd-normal space, $B \subseteq X$ be a countably compact subspace. Then for every closed set $F \subseteq X$ there is a family $\pi = \{OF\}$ of neighborhoods of F such that $|\pi| \le |B|$, $(\bigcap\{[OF] : OF \in \pi\} \cap [B]) \setminus F \cap [B]$ is discrete; moreover, if $F \cap B = \emptyset$, then $\bigcap\{[OF] : OF \in \pi\} \cap [B] = \emptyset$.*

Proof: We consider $F \cap [B]$. It is clear that there is a family $\pi = \{OF\}$ of neighborhoods of F such that $|\pi| = |B|$, $\bigcap\{[OF] : OF \in \pi\} \cap B = F \cap B$. Then $(\bigcap\{[OF] : OF \in \pi\} \cap [B]) \setminus F \cap [B]$ is discrete by Lemma 1.9. If $F \cap B = \emptyset$, then $\bigcap\{[OF] : OF \in \pi\} \cap [B] \subseteq [B] \setminus B$. In the same way as in the proof of Proposition 1.7 we can prove that $\bigcap\{[OF] : OF \in \pi\} \cap [B]$ is finite. We can add a finite number of neighborhoods of F to the family π and get what we need. The lemma is proved. □

Proposition 1.11. *Let X be an hd-normal, countably compact space such that every point $x \in X$ is a limit point of a closed set of cardinality at most $d(X)^{t(X)}$. Then $|X| \le d(X)^{t(X)}$.*

Proof: By Proposition 1.8 there is a dense, countably compact space $B \subseteq X$ such that every closed set $F \subseteq X \setminus B$ is finite and $|B| \le d(X)^{t(X)}$. Let $x \in X \setminus B$. There is a closed $F_x \subseteq X$ such that $|F_x| \le d(X)^{t(X)}$ and x is a limit point of F_x. Then by Lemma 1.9, $x \in [F_x \cap B]$. There is $F'_x \subseteq F_x \cap B$ such that $|F'_x| \le t(X)$ and $x \in [F'_x]$. Then $|X \setminus B| \le |B|^{t(X)} \le d(X)^{t(X)}$. The proposition is proved. □

Proposition 1.12. *Let X be an hd-normal compact space. Then $hl(X) \le s(X)^\omega$.*

Proof: We prove that $\chi(F, X) \le s(X)^\omega$ for every closed $F \subseteq X$. Really, for a closed $F \subseteq X$ there is a family $\pi = \{OF\}$ of neighborhoods of F such that $|\pi| \le s(X)$ and $d(\bigcap\{[OF] : OF \in \pi\} \setminus F) \le s(X)$ (this is well known, see for example [5]). By Proposition 1.8 there is a set $B \subseteq \bigcap\{[OF] : OF \in \pi\}$ such that $|B| \leq s(X)^\omega$, B is countably compact, $[B] \supseteq (\bigcap\{[OF] : OF \in \pi\}) \setminus F$ and every subset of B closed in X is finite. By Lemma 1.9 there is a family $\pi' = \{UF\}$, $|\pi'| \leq s(X)^\omega$ of neighborhoods of F such that $(\bigcap\{[UF] : UF \in \pi\} \cap [B]) \setminus F \cap [B]$ is discrete and therefore has cardinality at most $s(X)$. Finally, $\chi(F, X) \leq s(X)^\omega \cdot s(X) = s(X)^\omega$. The proposition is proved. $\square$

Recall that a free sequence of cardinality τ is a set $\xi\{x_\alpha : \alpha < \tau\}$ such that for all $\beta < \tau$ $[\{x_\alpha : \alpha < \beta\}] \cap [\{x_\alpha : \alpha \geq \beta\}] = \emptyset$ (see [6]).

Define $A(X) = \sup\{\tau : \tau$ is cardinality of a free sequence in $X\}$, $\varrho A(x, X) = A(X \setminus \{x\})$, $\varrho A(X) = \sup\{\varrho A(x, X) : x \in X\}$. A. Arhangelskii proved that $t(X) = A(X)$ for compact spaces [7]; moreover,

$$t(X) = A(X) \leq \varrho A(X) \leq s(X).$$

Note that for Alexandroff's double circle $s(X) = 2^\omega$, $\varrho A(X) = A(X) = \omega$. The same construction with Fedorchuk's compact space gives the space with $s(X) = 2^c$ and $\varrho A(X) = \omega$.

Theorem 1.13. *Let X be an hd-normal compact space. Then $\chi(x, X) \leq \varrho A(x, X)^\omega$.*

Proof: Let there be a point $x \in X$ such that $\varrho A(x, X)^\omega < \chi(x, X)$. Define $\varrho A(x, X) = \tau$. By induction we construct a set $D = \{y_\alpha : \alpha < \omega_{\tau^+}\}$, a family $\{B_\alpha : \alpha < \omega_{\tau^+}\}$, $|B_\alpha| \leq \tau^\omega$ of neighborhoods of x such that $([\{y_\alpha : \alpha < \delta\}] \cap \bigcap\{[Ox] : Ox \in B_\delta\}) \setminus \{x\} = \emptyset$, $\delta < \omega_{\tau^+}$. Let $y_0 \in X$, $B_0 = \{Ox\}$, where $[Ox] \not\ni y_0$. Let $\{y_\alpha : \alpha < \delta\}$ and $\{B_\alpha : \alpha < \delta\}$ be constructed. If $x \notin [\{y_\alpha : \alpha, \delta\}]$, let $B_\delta = \bigcup\{B_\alpha : \alpha < \delta\} \cup \{Ox\}$, where $Ox \cap \{y_\alpha : \alpha < \delta\} = \emptyset$. We choose y_δ in the set $\bigcap\{[Ox] : Ox \in B_\delta\} \setminus \{x\}$. If $x \in [\{y_\alpha : \alpha < \delta\}]$, we use Proposition 1.6 and Lemma 1.9. Since $|\{y_\alpha : \alpha < \delta\}| \leq \tau$, let us consider a family $\pi = \{Ox\}$ of neighborhoods of x, $|\pi| \leq \tau^\omega$ such that $T = (\bigcap\{[Ox] : Ox \in \pi\}) \cap [\{y_\alpha : \alpha < \delta\}] \setminus \{x\}$ is empty or is a free sequence in $X \setminus \{x\}$, and $|T| \leq \varrho A(x, X)$. Hence, there is a family π' of neighborhoods of x of cardinality at most τ^ω such that $(\bigcap\{[Ox] : Ox \in \pi\} \cap [\{y_\alpha : \alpha < \delta\}]) \setminus \{x\} = \emptyset$. Let $B_\delta \bigcup\{B_\alpha : \alpha < \delta\} \cup \pi'$, and choose y_δ from $\bigcap\{[Ox] : Ox \in B_\delta\} \setminus \{x\}$. If we continue until ω_{τ^+}, we get a free sequence of cardinality τ^+. But this contradicts $\varrho A(x, X) = \tau$. The theorem is proved. $\square$

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