

## Opial’s property and James’ quasi-reflexive spaces

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*Abstract.* Two of James’ three quasi-reflexive spaces, as well as the James Tree, have the uniform  $w^*$ -Opial property.

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### Introduction

Let  $(X, \|\cdot\|)$  be a Banach space with a Schauder finite dimensional decomposition (FDD) [1], [20]. Define  $\beta_p((X, \|\cdot\|))$  for  $p \in [1, \infty)$  to be the infimum of the set of numbers  $\lambda$  such that

$$(1) \quad (\|x\|^p + \|y\|^p)^{1/p} \leq \lambda\|x + y\|$$

for every  $x$  and  $y$  in  $X$  with  $\text{supp}(x) < \text{supp}(y)$  (we use here the notation in [1], [15]). In [15] M.A. Khamsi proved the following result.

**Theorem A.** *Let  $(X, \|\cdot\|)$  be a Banach space with a finite codimensional subspace  $Y$  such that  $\beta_p((Y, \|\cdot\|)) < 2^{1/p}$  for some  $p \in [0, \infty)$ . Then  $X$  has weak normal structure.*

He then used this theorem to deduce that James quasi-reflexive space, which consists of all null sequences  $x = \{x^i\} = \sum_{i=1}^\infty e_i$  ( $\{e_i\}$  is the standard basis in  $c_0$ ) for which the squared variation

$$(2) \quad \sup_{p_1 < \dots < p_m} \left[ \sum_{j=2}^m |x^{p_j} - x^{p_{j-1}}|^2 \right]^{1/2}$$

is finite, with the norm  $\|\cdot\|$  given by (2), has weak normal structure by claiming that  $\beta_2((J, \|\cdot\|_1)) = 1$ . As a matter of fact,  $\beta_2((J, \|\cdot\|_1)) \geq 2^{1/2}$ , which can be easily seen by taking  $x = e_2$  and  $y = e_3$ . Fortunately, Theorem A remains true with a slight modification of the definition of  $\beta_p((X, \|\cdot\|))$ , namely

$$\begin{aligned} & \tilde{\beta}((X, \|\cdot\|)) = \\ & = \inf_{k=0,1,2,\dots} \left\{ \inf[\lambda : (1) \text{ is valid for } x, y \in X \text{ with } \text{supp}(x) + k < \text{supp}(y)] \right\}, \end{aligned}$$

and  $\tilde{\beta}_2((J, \|\cdot\|)) = 1$ . Thus  $(J, \|\cdot\|_1)$  does indeed possess weak normal structure. In the present paper we will prove that  $(J, \|\cdot\|_1)$  has, in fact, the uniform  $w^*$ -Opial property [23], which, of course, also implies weak normal structure [1], [3], [4], [8].

**Definitions and notations**

We recall that a (dual) Banach space  $(X, \|\cdot\|)$  has the  $(w^*)$ -Opial property if whenever a sequence  $\{x_n\}$  in  $X$  converges weakly (weakly $^*$ ) to  $x_0$ , then for  $x \neq x_0$

$$\liminf_n \|x_0 - x_n\| < \liminf \|x - x_n\|$$

[21], [22]. Opial’s property plays an important role in the study of weak convergence of iterates and random products of nonexpansive mappings and of the asymptotic behavior of nonlinear semigroups [4], [5], [7], [13], [18], [21], [22]. Moreover, it can be introduced in the open unit ball of a complex Hilbert space, equipped with the hyperbolic metric, where it is useful in proving the existence of fixed points of holomorphic self-mappings of  $B$  [5], [6].

The (dual) Banach space  $(X, \|\cdot\|)$  is said to have the uniform  $(w^*)$ -Opial property [23] if for every  $c > 0$  there exists an  $r > 0$  such that

$$(3) \quad 1 + r \leq \liminf_n \|x - x_n\|$$

for each  $x \in X$  with  $\|x\| \geq c$  and every sequence  $\{x_n\}$  with  $w\text{-}\lim_n x_n = 0$  ( $w^*\text{-}\lim_n x_n = 0$ ) and  $\liminf_n \|x_n\| \geq 1$ .

In the linear space  $J$  defined by (2) one uses three different, but equivalent norms,  $\|\cdot\|_1$  (defined by (2)),  $\|\cdot\|_2$ , and  $\|\cdot\|_3$ , introduced by R.C. James [9], [10], [11]:

$$\|x\|_2 = \sup_{p_1 \dots < p_{2k}}^k \left[ \sum_{j=1}^k |x^{p_{2j}} - x^{p_{2j-1}}|^2 \right]^{1/2},$$

$$\|x\|_3 = \sup_{p_1 \dots < p_m}^m \left[ \sum_{j=2}^m |x^{p_j} - x^{p_{j-1}}|^2 + |x^{p_m} - x^{p_1}|^2 \right]^{1/2}.$$

The choice of norms depends on one’s goals [2], [9], [10], [11], [20].

In [14] M.A. Khamsi used the ultraproduct method to prove that  $(J, \|\cdot\|_3)$  has the fixed point property for nonexpansive mappings (FPP), i.e. for every nonempty weakly compact convex subset  $C$  of  $(J, \|\cdot\|_3)$  any nonexpansive self-mapping  $T : C \rightarrow C$  has a fixed point. D. Tingley [24] has recently shown that  $(J, \|\cdot\|_3)$  has, in fact, weak normal structure ([3]): every nonempty weakly compact convex subset  $C$  of  $(J, \|\cdot\|_3)$  with  $\text{diam } C > 0$  has a nondiametral point  $y$ , i.e.

$$\sup_{x \in C} \|y - x\|_3 < \text{diam } C.$$

This property immediately guarantees the FPP [17]. The proof of weak normal structure is based on the following property of weakly convergent sequences in  $(J, \|\cdot\|_3)$ : if  $\{x_n\}$  converges weakly to 0 and  $\text{diam } \{x_n\} > 0$ , then

$$\sup_m \left( \limsup_n \|x_m - x_n\|_3 \right) > \liminf_n \|x_n\|_3.$$

But it is easy to observe that the sequence  $\{-e_n + e_{n+1}\}$  tends weakly to 0 in  $(J, \|\cdot\|_3)$  and

$$\left\| \frac{1}{3}e_1 + e_n - e_{n+1} \right\|_3 = \left\| -e_n + e_{n+1} \right\|_3 = \sqrt{8}$$

for  $n \geq 3$ . Therefore  $(J, \|\cdot\|_3)$  does not have Opial's property.

### Main result

In this section we are concerned with the spaces  $(J, \|\cdot\|_1)$  and  $(J, \|\cdot\|_2)$ .

The predual Banach space  $I$  to  $(J, \|\cdot\|_j)$ ,  $j = 1, 2$ , is generated by the biorthogonal functionals  $\{f_n\}$  to the basis  $\{u_n\} = \{e_1 + \dots + e_n\}$  [12], [19]. Throughout this paper we will always treat  $J$  as  $I^*$ .

**Theorem.** For  $j = 1, 2$  the space  $(J, \|\cdot\|_j)$  has the uniform  $w^*$ -Opial property.

PROOF: Let  $k \in \mathbb{N}$  and let  $P_k$  and  $Q_k$  be the natural projections in  $J$  associated with the basis  $\{u_n\}$ :

$$P_k x = \sum_{n=1}^k \xi^n u_n$$

and

$$Q_k x = \sum_{n=k+1}^{\infty} \xi^n u_n$$

for each  $x = \sum_{n=1}^{\infty} \xi^n u_n \in J$ . Note that if  $x = \sum_{n=1}^{\infty} \xi^n u_n$ , then

$$\|x\|_1 = \sup_{p_1 < \dots < p_m} \left\{ \sum_{j=2}^m \left[ \sum_{n=p_{j-1}}^{p_j-1} \xi^n \right]^2 \right\}^{1/2}$$

and

$$\|x\|_2 = \sup_{p_1 < \dots < p_{2k}} \left\{ \sum_{j=1}^k \left[ \sum_{n=p_{2j-1}}^{p_{2j}-1} \xi^n \right]^2 \right\}^{1/2}$$

[11], [12]. Directly from these formulas we obtain

$$\|x\|_j = \lim_k \|P_k x\|_j$$

and

$$\lim_k \|Q_k x\|_j = 0$$

for all  $x \in J$  and  $j = 1, 2$ . Assume that a sequence  $\{x_n\}$  in  $(J, \|\cdot\|_j)$  converges weakly\* to 0 and let  $x \in J$ . Then we have

$$\begin{aligned} \lim_n \|P_k x_n\|_j &= 0, \\ \liminf_n \|Q_k x_n\|_j &= \liminf_n \|x_n\|_j, \end{aligned}$$

and

$$\begin{aligned} \liminf_n \|x - x_n\| &\geq \liminf_n \left[ \|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j - \|P_{k+1} x_n\|_j \right] \\ &= \liminf_n \left[ \|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j \right] \\ &\geq \liminf_n \left[ \|P_k x\|_j^2 + \|Q_{k+1} x_n\|_j^2 \right]^{1/2} - \|Q_k x\|_j \\ &= \left[ \|P_k x\|_j^2 + \liminf_n \|Q_{k+1} x_n\|_j^2 \right]^{1/2} - \|Q_k x\|_j \\ &= \left[ \|P_k x\|_j^2 + \liminf_n \|x_n\|_j^2 \right]^{1/2} - \|Q_k x\|_j \end{aligned}$$

for  $k = 1, 2, \dots$ . Hence we obtain the following inequality

$$\begin{aligned} (*) \quad \liminf_n \|x - x_n\|_j &\geq \lim_k \left\{ \left[ \|P_k x\|_j^2 + \liminf_n \|x_n\|_j^2 \right]^{1/2} - \|Q_k x\|_j \right\} \\ &= \left[ \|x\|_j^2 + \liminf_n \|x_n\|_j^2 \right]^{1/2} \end{aligned}$$

which leads to (3). In other words,  $(J, \|\cdot\|_j)$  has the uniform  $w^*$ -Opial property for  $j = 1, 2$ . □

**Corollary 1.** *For  $j = 1, 2$  the space  $(J, \|\cdot\|_j)$  has the uniform Opial property.*

**Remark 1.** The uniform  $w^*$ -Opial property of  $(J, \|\cdot\|_j)$ ,  $j = 1, 2$ , implies the following important property of these spaces. The  $(w^*)$ -modulus of noncompact convexity of a (dual) Banach space  $(X, \|\cdot\|)$  is the function  $\Delta_x : [0, 1] \rightarrow [0, 1]$  ( $\Delta_x^* : [0, 1] \rightarrow [0, 1]$ ) defined by

$$\begin{aligned} \Delta_x(\varepsilon) &= \inf\{1 - \text{dist}(0, A)\} \\ (\Delta_x^*(\varepsilon) &= \inf\{1 - \text{dist}(0, A)\}), \end{aligned}$$

where the infimum is taken over all convex (weak\* compact convex) subsets  $A$  of the closed unit ball with  $\chi(A) \geq \varepsilon$ , and  $\chi$  is the Hausdorff measure of noncompactness [4]. In the case of  $(J, \|\cdot\|_j)$ ,  $j = 1, 2$ , the inequality (\*) implies  $\Delta_j^*(\varepsilon) > 0$

for all  $\varepsilon > 0$ . This means that these spaces are  $\Delta^*$ -uniformly convex and every weakly\* compact convex subset  $C$  of  $(J, \|\cdot\|_j)$  ( $j = 1, 2$ ) has a compact asymptotic center [4]. Taking  $A = \text{conv}\{u_n\}$ , where  $u_n = \sum_{i=1}^n$ , we see that

$$\chi(A) = 1$$

and

$$\forall_{x \in A} \|x_j\| = 1$$

( $j = 1, 2$ ). Therefore  $\Delta_x \equiv 0$  for  $X = (J, \|\cdot\|_j)$ ,  $j = 1, 2$ .

Here we have to mention that generally the uniform Opial property does not imply the  $\Delta$ -uniform convexity as the following example shows.

**Example** ([23]). For  $\lambda > 1$  let  $X$  be the space  $l_2$  with the norm

$$\|(\alpha_n)\| = \max\{\lambda|\alpha_1|, \|(\alpha_n)\|_2\}$$

where  $\|\cdot\|_2$  is the norm in  $l_2$ . Then

$$\begin{aligned} \liminf_n \|x_n - x\| &= \max\left\{\lambda|\alpha_1|, \left(\liminf_n \|x_n\|_2^2 + \|x\|_2^2\right)^{1/2}\right\} \\ &\geq (1 + \|x\|_2^2)^{1/2} \geq (1 + \lambda^{-2}\|x\|_2^2)^{1/2} \end{aligned}$$

for each  $x \in X$  and each sequence  $\{x_n\}$  with  $w\text{-}\lim_n x_n = 0$  and  $\liminf \|x_n\| \geq 1$ . This inequality guarantees the uniform Opial property of  $X$ , but  $\Delta_x(\varepsilon) = 0$  for all  $\varepsilon \leq (1 - \lambda^{-2})^{1/2}$ .

**Remark 2.** It is easy to observe that James Tree  $JT$  constructed by R.C. James [11] also has the  $w^*$ -uniform Opial property, where  $JT$  is the dual space to the Banach space  $B$  generated by the biorthogonal functionals  $\{f_{n,i}\}$  to the basis  $\{e_{n,i}\}$  (this basis is analogous to the basis  $\{u_n\}$  in  $J$ ) given in [19]. The proof of this fact is a slight modification of the proof of the Theorem. Corollary 1 and Remark 1 are also valid for  $JT$ . (See also [13] for the  $w^*$ -Opial property.)

**Remark 3.** The uniform ( $w^*$ -) Opial property of  $(J, \|\cdot\|_j)$  with  $j = 1, 2$  and  $JT$  implies that these spaces satisfy the weak (weak\*) uniform Kadec-Klee property [16].

We conclude our paper with three corollaries.

**Corollary 2.**  $(J, \|\cdot\|_j)$ ,  $j = 1, 2$ , and  $JT$  have weak and weak\* normal structure.

**Corollary 3.**  $(J, \|\cdot\|_j)$ ,  $j = 1, 2$ , and  $JT$  have the FPP for weakly\* compact convex subsets.

Recall that a Banach space  $(X, \|\cdot\|)$  is said to satisfy the ( $w^*$ -) demiclosedness principle [1], [4] if whenever  $C$  is a nonempty weakly (weakly\*) compact convex subset of  $X$  and  $T : C \rightarrow X$  is nonexpansive, then the mapping  $I - T$ , where  $I$  is the identity operator, is ( $w^*$ -) demiclosed, i.e. if  $\{x_n\}$  is weakly (weakly\*) convergent to  $x$  and  $\{x_n - Tx_n\}$  converges strongly to  $y$ , then  $x - Tx = y$ . It is known that every Banach space with the ( $w^*$ -) Opial property satisfies the ( $w^*$ -) demiclosedness principle.

**Corollary 4.**  $(J, \|\cdot\|_j)$ ,  $j = 1, 2$ , and  $JT$  satisfy the  $(w^*)$ -demiclosedness principle.

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