The area formula for $W^{1,n}$-mappings

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Abstract. Let f be a mapping in the Sobolev space $W^{1,n}(\Omega, \mathbf{R}^n)$. Then the change of variables, or area formula holds for f provided removing from counting into the multiplicity function the set where f is not approximately Hölder continuous. This exceptional set has Hausdorff dimension zero.

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1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open set. Le $f\colon \Omega \to \mathbf{R}^n$ be a mapping and $S \subset \Omega$. We define the multiplicity function (Banach indicatrix) $\mathcal{N}$ by

$$\mathcal{N}(y, f, S) = \sharp\{x \in S\colon f(x) = y\}\,.$$

If f is a Lipschitz mapping on Ω, then f is differentiable a.e. (Rademacher theorem) and the area formula

$$\int_S |\det \nabla f(x)|\,dx = \int_{\mathbf{R}^n} \mathcal{N}(y, f, S)\,dy \tag{1.1}$$

holds for any measurable set $S \subset \Omega$ (see [2]). The same is true if f is a continuous representative of a mapping in $W^{1,p}(\Omega, \mathbf{R}^n)$ with $p > n$ (as the Lusin (N)-property holds, cf. Proposition 1.1 and [1]). There are continuous mappings in $W^{1,p}(\Omega, \mathbf{R}^n)$ with $p \le n$ for which the area formula does not hold (see [13], [9] and references therein). The problem of the area formula for Sobolev mappings is continuously stimulating. For interesting recent results we refer to [10]. One approach consists in looking for "partial area formulae": a set S_0 of full measure is found such that

$$\int_S |\det \nabla f(x)|\,dx = \int_{\mathbf{R}^n} \mathcal{N}(y, f, S \cap S_0)\,dy$$

for all measurable $S \subset \Omega$.

As shown by Federer [3], for any f which has partial derivatives almost everywhere there are sequences f_j of Lipschitz mappings and M_j of disjoint measurable sets such that $f_j = f$ on M_j and $\Omega \setminus \bigcup_j M_j$ has zero measure. The following proposition is then an easy and well known consequence (cf. [10], [5]).

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1.1 Proposition. *Let $S \subset \Omega$ be a measurable set and $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$. Then the following assertions are equivalent:*

(i) *"the (N)-property holds for f on S": $|f(E)| = 0$ for each $E \subset S$ with $|E| = 0$,*

(ii) *"the area formula holds for f on S":*

$$\int_{S'} |\det \nabla f(x)|\, dx = \int_{\mathbf{R}^n} \mathcal{N}(y, f, S')\, dy$$

for each measurable set $S' \subset S$,

(iii) *"the change of variables formula holds for f on S":*

$$\int_{S} u(f(x))|\det \nabla f(x)|\, dx = \int_{\mathbf{R}^n} u(y)\, \mathcal{N}(y, f, S)\, dy$$

for each nonnegative Borel measurable function u on $\mathbf{R}^n$.

Let f be a function in L^1_{loc} defined a.e. in Ω. Then the function

$$\tilde{f}(x) := \lim_{r \to 0+} \fint_{B(x,r)} f\,.$$

is defined in Ω except a set of measure zero. The function $\tilde{f}$ is called the *Lebesgue representative* of f and we say that f is *Lebesgue precise* if $f = \tilde{f}$. If, in addition, $f \in W^{1,p}(\Omega)$ with $p > 1$, then $\tilde{f}$ is defined up to a set of p-capacity zero and p-finely continuous except for a set of p-capacity zero (see [14, Section 3.3]), which means that it is p-quasi-continuous ([6, Theorem 8]). These references are also recommended for definitions of p-capacity, p-quasi-continuity and p-fine topology (by the p-capacity cap_p we understand the Bessel capacity denoted by $B_{1,p}$ in [14]).

The following theorem gives a good choice of a set of canonical nature for which the area formula holds ([5], cf. also [4]).

1.2 Proposition. *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ be Lebesgue precise and S be the set of all points of Ω at which f is approximately differentiable. Then the area formula holds for f on S.*

The aim of this paper is to use a slight refinement of methods from [9] to show that for $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ the set to be removed for validity of the area formula can be found even smaller. The following theorem will be proved in the next section.

1.3 Theorem. *Suppose that f is an n-quasi-continuous representative of a mapping in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and S is the set of all points of Ω at which f is approximately Hölder continuous. Then the area formula holds for f on S.*

The size of the exceptional set is estimated in the following result, proved in Section 3.

1.4 Theorem. *Suppose that f is an n-quasi-continuous representative of a mapping in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and S is the set of all points of Ω at which f is approximately Hölder continuous. Then the set $\Omega \setminus S$ has Hausdorff dimension zero.*

2. Points of approximate Hölder continuity

Let f be a measurable function on Ω. We say that f is *approximately Hölder continuous* at $x \in \Omega$ if there is $\alpha \in (0,1]$ and a set M such that

$$\limsup_{y\to x,\, y\in M} \frac{|f(y)-f(x)|}{|y-x|^\alpha} < \infty$$

and the Lebesgue density of M at x is one.

We need the following version of the Gehring oscillation lemma.

2.1 Lemma. *Let f be a quasi-continuous representative of a mapping in $W^{1,n}(B(x,r), \mathbf{R}^m)$. Then for almost all $t \in (0,r)$ the restriction of f to $\partial B(x,t)$ is a continuous representative of an element of $W^{1,n}(\partial B(x,t), \mathbf{R}^m)$ and the inequality*

$$\Big(\operatorname{diam} f(\partial B(x,t))\Big)^n \le c\, t \int_{\partial B(x,t)} |\nabla f|^n dS\,. \tag{2.1}$$

holds.

Proof: The estimate follows from the Sobolev inequality and a similarity argument if f is $\mathcal{C}^1$. In the general case there are $\mathcal{C}^1$ mappings f_j such that

$$\sum_j \|f_j - f\|^p_{1,p} < \infty\,.$$

Using integration over radii it follows that there is $N_1 \subset (0,r)$ with $|N_1| = 0$ such that

$$\sum_j \int_{\partial B(x,t)} (|f_j - f|^p + |\nabla f_j - \nabla f|^p)\, dS < \infty\,.$$

Since f is n-quasi-continuous and $f_j \to f$ in $W^{1,n}$, we know (after selecting a subsequence) that $f_j \to f$ except a set E of n-capacity zero ([12, Theorem 5.4]). By well known relations between capacity and Hausdorff measure ([12]) it follows that the linear measure of E is zero, so that there is $N_2 \subset (0,r)$ with $|N_2| = 0$ such that $f_j \to f$ everywhere on $\partial B(x,t)$ for each $t \in (0,r)\setminus N_2$. If $t \in (0,r)\setminus(N_1 \cup N_2)$, then the Sobolev inequality implies uniform convergence $f_j \to f$ on $\partial B(x,t)$ and a routine passage to limit yields (2.1). □

The following tool is essentially Lemma 4.5 of [9].

2.2 Lemma. *Let B be a ball $B(x,r)$ and $\tau \in (0,1)$. Suppose $f \in W^{1,n}(B,\mathbf{R}^n)$. Then there is a measurable set $A \subset B$ such that*

$$|B \setminus A| \le \tau |B|$$

and

$$(\operatorname{diam} f(A))^n \le c \int_A (1+|\nabla f|^n)\, dy\,,$$

c depends only on n and τ.

Proof: Although the proof in fact follows the idea in [9], we present it here, as in the original the mapping f is assumed to be continuous. Set

$$\omega_0 = \inf\{\operatorname{diam} f(\partial B(x,t))\colon t \in [r/2,r]\}\,.$$

Find $t_0 \in [r/2,r]$ such that $\operatorname{diam} f(\partial B(x,t_0)) < \omega_0 + r$ and choose $z_0 \in \partial B(x,t_0)$. Denote

$$\begin{aligned} &u(y) = |f(y) - f(z_0)|\,,\\ &\lambda_i = i(r+\omega_0)\,,\\ &E_i = \{y \in B(x,r)\colon |f(y)-f(z_0)| < \lambda_i\}\,,\\ &v_{i,j} = \max(\min(u,\lambda_j),\lambda_i)\,. \end{aligned}$$

First we will derive the estimate

$$\omega_0^n \le c \int_{B(x,r)\cap E_3} |\nabla f|^n\, dy\,. \tag{2.2}$$

If there exists $t_1 \in [r/2,r]$ such that $\partial B(x,t_1) \cap E_2 = \emptyset$, we observe that $v_{1,2} = \lambda_1$ on $\partial B(x,t_0)$ and $v_{1,2} = \lambda_2$ on $\partial B(x,t_1)$. It follows

$$(r+\omega_0)^n = (\lambda_2 - \lambda_1)^n \le c \int_{B(x,r)} |\nabla v_{1,2}|^n\, dy \le c \int_{E_3 \cap B(x,r)} |\nabla f|^n\, dy\,,$$

which proves (2.2) in this case. Now we may assume that $\partial B(x,t)$ intersects E_2 for all $t \in [r/2,r]$. We write $F = \{t \in [r/2,r] : \partial B(x,t) \subset E_3\}$. Using Lemma 2.1 to f and $v_{2,3}$, for almost all $t \in [r/2,r]$ we get

$$t^{-1}\omega_0^n \le c \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n\, dS\,. \tag{2.3}$$

Indeed, if $t \in F$, we estimate

$$\begin{aligned} \omega_0^n &\le (\operatorname{diam} f(\partial B(x,t)))^n \le c\,t \int_{\partial B(x,t)} |\nabla f|^n\, dS\\ &= c\,t \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n\, dS\,, \end{aligned}$$

while for $t \in [r/2, r] \setminus F$ we have

$$\begin{aligned}\omega_0^n \le (\lambda_3 - \lambda_2)^n \le (\operatorname{diam} v_{2,3}(\partial B(x,t)))^n \le c\, t \int_{\partial B(x,t)} |\nabla v_{2,3}|^n \, dS \\ \le c\, t \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n \, dS \,.\end{aligned}$$

Integrating (2.3) over $t \in [r/2, r]$ we obtain (2.2). Now, if $|B \setminus E_3| \le \tau |B|$, we are done. Indeed, setting $A = B \cap E_3$, we obtain

$$(\operatorname{diam} f(A))^n \le c(\omega_0 + r)^n \le c \int_A |\nabla f|^n \, dy + c r^n \le c \int_A (1 + |\nabla f|^n) \, dy \,.$$

Otherwise we find $k \ge 3$ such that

$$|B \setminus E_{k+1}| \le \tau |B| < |B \setminus E_k|$$

and set $A = B \cap E_{k+1}$. Since $v_{1,k} - \lambda_1 = 0$ on $\partial B(x, t_0)$, a Poincaré-type inequality ([14, Section 4.5]) yields

$$\begin{aligned}(\operatorname{diam} f(A))^n \le (2\lambda_{k+1})^n \le c\tau^{-1} r^{-n} \int_B (v_{1,k} - \lambda_1)^n \, dy \le c\tau^{-1} \int_B |\nabla v_{1,k}|^n \, dy \\ \le c\tau^{-1} \int_A |\nabla f|^n \, dy \,,\end{aligned}$$

which concludes the proof. □

Proof of Theorem 1.3: We verify the condition (i) of Proposition 1.1. Recall that S is the set of all points where f is approximately Hölder continuous. Let $E \subset S$ be a set of zero measure. Decomposing E if necessary into a countable union, we may assume that f is $1/m$-Hölder continuous at all $x \in E$ for $m \in \mathbf{N}$ fixed. Choose an open set $G \subset \Omega$ containing E. Fix $x \in E$. There are $K > 0$ and a set $M \subset \Omega$, such that the Lebesgue density of M at x is one and

$$|f(y) - f(x)| \le K |y - x|^{1/m}$$

for all $y \in M$. Find $r_0 > 0$ such that $B(x, r_0) \subset G$ and

$$|B(x,r) \setminus M| \le \frac{1}{4} |B(x,r)|$$

for all $r \in (0, r_0)$. For $k = 0, 1, \dots$ we denote

$$\begin{aligned} r_k &= r_0 2^{-k}, \\ B_k &= B(x, r_k) \end{aligned}$$

and using Lemma 2.2 we find a measurable set $A_k \subset B_k$ such that

$$|B_k \setminus A_k| \le 2^{-n-2}|B_k|$$

and

$$(\operatorname{diam} f(A_k))^n \le c \int_{A_k} (1 + |\nabla f|^n)\, dy\,. \tag{2.4}$$

Then (for $k \ge 1$)

$$\begin{aligned} &|B_k \setminus A_{k-1}| \le 2^{-n-2}|B_{k-1}| = \frac{1}{4}|B_k|,\\ &|B_k \setminus A_k| \le 2^{-n-2}|B_k|,\\ &|B_k \setminus M| \le \frac{1}{4}|B_k|, \end{aligned}$$

and thus there is $x_k \in A_k \cap A_{k-1} \cap M$, $x_k \ne x$. We have

$$|f(x_k) - f(x)| \le c|x_k - x|^{1/m} \le c2^{-k/m}\,. \tag{2.5}$$

Choosing b with $(1 + 1/b)^{-1} > 2^{-1/m}$ we claim that the set

$$I(x) = \{k \in \mathbf{N} \colon |f(x_{k+1}) - f(x)| \le b|f(x_k) - f(x_{k+1})|\}$$

is infinite. Indeed, assuming that $\max I(x) = k_0$, we get

$$|f(x_k) - f(x)| \le |f(x_{k+1}) - f(x)| + |f(x_k) - f(x_{k+1})| \le (1 + 1/b)|f(x_{k+1}) - f(x)|$$

for each $k > k_0$, which leads to a contradiction with (2.5), as an iteration yields

$$|f(x_k) - f(x)| \ge c(1 + 1/b)^{-k}\,.$$

Denote

$$R_k(x) = \operatorname{diam} f(A_k) + |f(x_{k+1}) - f(x)|\,.$$

Since $x_k, x_{k+1} \in A_k$, we have

$$f(A_k) \subset B(f(x), R_k)$$

and (using (2.4))

$$\begin{aligned} R_k(x) &\le \operatorname{diam} f(A_k) + b|f(x_{k+1}) - f(x_k)| \le (1 + b)\operatorname{diam} f(A_k)\\ &\le c(1 + b)\Big(\int_{A_k} (1 + |\nabla f|^n)\, dy\Big)^{1/n} \end{aligned}$$

whenever $k \in I(x)$. For the next step of the proof we write this estimate in the form

$$(2.6) \qquad (R_k(x))^n \le c \int_{f^{-1}(B(f(x),R_k(x)))} (1+|\nabla f|^n)\, dy\,, \qquad k \in I(x)\,.$$

The balls $B(f(x),R_k)$, $x \in E$, $k \in I(x)$, form a Vitali cover of $f(E)$. By the Vitali covering theorem there is a disjoint subcover $B(f(x_\tau),R_\tau)$ such that the set $N := E \setminus \bigcup_\tau B(f(x_\tau),R_\tau)$ has zero Lebesgue measure. Using (2.6) it follows that

$$\begin{aligned} |f(E)| &\le |N| + 2^n \sum_\tau R_\tau^n \\ &\le c \sum_\tau \int_{f^{-1}(B(f(x_\tau),R_\tau^n))} (1+|\nabla f|^n)\, dy \le c \int_G (1+|\nabla f|^n)\, dy\,. \end{aligned}$$

Varying the set G we get the required conclusion $|f(E)| = 0$. □

3. Size of the exceptional set

3.1 Lemma. *Let $p > 1$. Then the p-fine closure of any open set $A \subset \mathbf{R}^n$ has the same p-capacity as A.*

Proof: See [11, Proposition 3.2]. □

The following tool is an immediate consequence of Theorem 7 of [8].

3.2 Proposition. *Let f be an n-quasi-continuous representative of a mapping in $W^{1,n}(\Omega,\mathbf{R}^n)$, $\varepsilon > 0$ and $p < n$. Then there is an open set $G \subset \mathbf{R}^n$ such that $\operatorname{cap}_p G < \varepsilon$ and the restriction of f to $\Omega \setminus G$ is locally Hölder continuous.*

Proof of Theorem 1.4: Using Proposition 3.2 we find open sets $G_{i,j}$ and $p_i \nearrow n$ such that $\operatorname{cap}_{p_i}(G_{i,j}) < 1/j$ and f is locally Hölder continuous on $\Omega \setminus G_{i,j}$. Denote by $S_{i,j}$ the p_i-fine interior of $\mathbf{R}^n \setminus G_{i,j}$ and $S = \Omega \cap \bigcup_{i,j} S_{i,j}$. By Lemma 3.1, $\operatorname{cap}_{p_i}(\mathbf{R}^n \setminus S_{i,j}) < 1/j$, and thus the Hausdorff dimension of E is zero (see e.g. [7, Theorem 2.26]). Consider a point $x \in S$. Then there are i,j such that $x \in \Omega \cap S_{i,j}$. Since the set $S_{i,j}$ is p_i-finely open and f is locally Hölder continuous on $S_{i,j}$, it follows that the Lebesgue density of S at x is one ([14, Section 3.3]) and f is approximately Hölder continuous at x. □

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