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Ergodic properties of contraction semigroups in L_p, $1 < p < \infty$

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Abstract. Let $\{T(t) : t > 0\}$ be a strongly continuous semigroup of linear contractions in L_p, $1 < p < \infty$, of a σ-finite measure space. In this paper we prove that if there corresponds to each $t > 0$ a positive linear contraction $P(t)$ in L_p such that $|T(t)f| \leq P(t)|f|$ for all $f \in L_p$, then there exists a strongly continuous semigroup $\{S(t) : t > 0\}$ of positive linear contractions in L_p such that $|T(t)f| \leq S(t)|f|$ for all $t > 0$ and $f \in L_p$. Using this and Akcoglu's dominated ergodic theorem for positive linear contractions in L_p, we also prove multiparameter pointwise ergodic and local ergodic theorems for such semigroups.

Keywords: contraction semigroup, semigroup modulus, majorant, pointwise ergodic theorem, pointwise local ergodic theorem

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1. Introduction and the main result

Let (X, Σ, μ) be a σ-finite measure space and let $L_p = L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, denote the usual Banach spaces of real or complex functions on (X, Σ, μ). A linear operator $T : L_p \to L_p$ is called a **contraction** if $\|T\|_p \leq 1$, $\|T\|_p$ being the operator norm of T in L_p, **positive** if $0 \leq f \in L_p$ implies $Tf \geq 0$, and **majorizable** if there exists a positive linear operator $P : L_p \to L_p$ such that $|Tf| \leq P|f|$ for all $f \in L_p$. Any such P will be referred to as a **majorant** of T. It is known (cf. [5, § 4.1]) that a bounded linear operator T in L_p possesses a majorant P when $p = 1$ or ∞. But this is not the case when $1 < p < \infty$. The Hilbert transform serves as an example in L_p for all $1 < p < \infty$ (see Starr [8]). The following proposition is needed later, whose proof is omitted because it is essentially the same as that of Theorem 4.1.1 in [5].

Proposition (cf. [5], Remark, p. 161)**.** *Let T be a bounded linear operator in L_p, $1 < p < \infty$, and let P be a majorant of T. Then there exists a unique positive linear operator τ in L_p, called the linear modulus of T, such that*

(i) $\|\tau\|_p \leq \|P\|_p$,
(ii) $|Tf| \leq \tau|f|$ *for all* $f \in L_p$,
(iii) $\tau f = \sup\{|Tg| : g \in L_p,\ |g| \leq f\}$ *for all* $f \in L_p^+$.

From now on let us fix p with $1 < p < \infty$. Let $\{T(t) : t > 0\}$ be a strongly continuous semigroup of linear contractions in L_p, i.e.

(i) each $T(t)$ is a linear contraction in L_p,
(ii) $T(t)T(s) = T(t+s)$ for all $t, s > 0$,
(iii) $\lim_{t \to s} \|T(t)f - T(s)f\|_p = 0$ for all $f \in L_p$ and $s > 0$.

Since the operators $T(t)$ are not necessarily majorizable, it cannot be expected that the semigroup $\{T(t) : t > 0\}$ is majorizable by a positive semigroup, i.e. there exists a strongly continuous semigroup $\{S(t) : t > 0\}$ of positive linear operators in L_p such that $|T(t)f| \leq S(t)|f|$ for all $t > 0$ and $f \in L_p$. But if each $T(t)$ possesses a majorant $P(t)$ such that $\|P(t)\|_p \leq 1$, then we can prove the following main result in this paper.

Theorem 1 (cf. Theorem 1 in [7]). *Let $\{T(t) : t > 0\}$ be a strongly continuous semigroup of linear contractions in L_p, $1 < p < \infty$. Suppose each $T(t)$ possesses a majorant $P(t)$ such that $\|P(t)\|_p \leq 1$. Then there exists a strongly continuous semigroup $\{S(t) : t > 0\}$ of positive linear contractions in L_p, called the semigroup modulus of $\{T(t) : t > 0\}$, such that*

(i) $|T(t)f| \leq S(t)|f|$ *for all* $t > 0$ *and* $f \in L_p$,
(ii) $S(t)f = \sup\{\tau(t_1)\dots\tau(t_n)f : \sum_{i=1}^n t_i = t,\ t_i > 0,\ n \geq 1\}$ *for all* $f \in L_p^+$, *where* $\tau(t)$ *denotes the linear modulus of* $T(t)$,
(iii) $\tau(0) = \text{strong-}\lim_{t \to +0} S(t)$, *where* $\tau(0)$ *denotes the linear modulus of* $T(0) = \text{strong-}\lim_{t \to +0} T(t)$.

Proof: For an $f \in L_p^+$ and $t > 0$, define

$$S(t)f = \sup\Big\{\tau(t_1)\dots\tau(t_n)f : \sum_{i=1}^n t_i = t,\ t_i > 0,\ n \geq 1\Big\}. \tag{1}$$

Since $\|\tau(t)\|_p \leq \|P(t)\|_p \leq 1$ and $\tau(t)\tau(s) \geq \tau(t+s) \geq 0$ for all $t, s > 0$, it follows that

$$\|S(t)f\|_p \leq \|f\|_p \tag{2}$$

and that

$$S(t)(cf) = cS(t)f \quad \text{and} \quad S(t)(f+g) = S(t)f + S(t)g \tag{3}$$

for a constant $c > 0$ and $f, g \in L_p^+$. Thus we may regard $S(t)$ as a positive linear contraction in L_p. From the definition of $S(t)$ it easily follows that

$$S(t)S(s) = S(t+s) \quad \text{for all} \quad t, s > 0. \tag{4}$$

Since (i) is clear, to complete the proof it is enough to establish (iii), because (iii) together with the fact that $\|S(t)\|_p \leq 1$ for all $t > 0$ implies that for every $f \in L_p$ and $s > 0$

$$\begin{aligned}\lim_{t \to +0} \|S(s)f - S(s+t)f\|_p &\leq \lim_{t \to +0} \|S(s-t)\|_p \|S(t)f - S(2t)f\|_p \\ &\leq \lim_{t \to +0} \big(\|S(t)f - \tau(0)f\|_p + \|S(2t)f - \tau(0)f\|_p\big) = 0,\end{aligned}$$

and similarly $\lim_{t\to+0}\|S(s)f - S(s-t)f\|_p = 0$; namely, $\{S(t) : t > 0\}$ is strongly continuous at each $s > 0$. For this purpose we first remark that $T(0) =$ strong-$\lim_{t\to+0}T(t)$ exists. This is due to Lemma 1 in [6], because L_p is a reflexive Banach space and $\|T(t)\|_p \le 1$ for all $t > 0$.

We next show that the linear modulus $\tau(0)$ of $T(0)$ exists. To do this, define

$$P(0)f = \sup\left\{|T(0)g| : g \in L_p,\ |g| \le f\right\} \quad \text{for} \quad f \in L_p^+ . \tag{5}$$

Since $\lim_{t\to+0}\|T(t)g - T(0)g\|_p = 0$, it follows that there exists a sequence $\{t_n\}$ of positive reals with $t_n \downarrow 0$ for which

$$T(0)g = \lim_n T(t_n)g \quad \text{a.e. on } X.$$

Then

$$|T(0)g| \le \liminf_n \tau(t_n)|g| \le \liminf_n \tau(t_n)f \quad \text{a.e. on } X.$$

Since there are countable functions $g_i \in L_p$, $1 \le p \le \infty$, such that $|g_i| \le f$ and $P(0)f = \sup_i |T(0)g_i|$ a.e. on X, we apply the Cantor diagonal argument to infer that there exists a sequence $\{t_n\}$ of positive reals with $t_n \downarrow 0$ for which

$$P(0)f \le \liminf_n \tau(t_n)f \quad \text{a.e. on } X.$$

Then, by Fatou's lemma,

$$\|P(0)f\|_p \le \liminf_n \|\tau(t_n)f\|_p \le \|f\|_p \quad (f \in L_p^+). \tag{6}$$

It also follows from the proof of Theorem 4.1.1 in [5] that if $\{B_1, \ldots, B_m\}$ is a finite measurable partition of X, then

$$\sum_{i=1}^{m} |T(0)(1_{B_i}f)| \le P(0)f \quad \text{a.e. on } X, \tag{7}$$

where 1_{B_i} denotes the indicator function of B_i. Thus we see, as in the proof of Theorem 4.1.1 in [5], that the linear modulus $\tau(0)$ of $T(0)$ exists. (Incidentally we note that $\tau(0)f = P(0)f$ for all $f \in L_p^+$.)

To prove (iii), let $f \in L_p^+$ be fixed arbitrarily, and given an $\varepsilon > 0$ choose $g_i \in L_p$, $1 \le i \le n$, so that

$$|g_i| \le f \quad \text{and} \quad \|\tau(0)f - \max_i |T(0)g_i|\|_p < \varepsilon.$$

Since $T(0) =$ strong-$\lim_{t\to+0}T(t)$, choose $\delta > 0$ so that

$$0 < t < \delta \ \text{ implies } \ \|T(0)g_i - T(t)g_i\|_p < \varepsilon/n \quad (1 \le i \le n).$$

Then, putting $h_0 = \max_i |T(0)g_i|$ and $h_t = \max_i |T(t)g_i|$ for $t > 0$, we get

$$|h_0 - h_t| \le \max_i |T(0)g_i - T(t)g_i| \le \sum_{i=1}^{n} |T(0)g_i - T(t)g_i|,$$

and hence $\|h_0 - h_t\|_p \le \sum_{i=1}^{n} \|T(0)g_i - T(t)g_i\|_p < \varepsilon$ for $0 < t < \delta$. Thus

$$\begin{aligned}\|\tau(0)f - \max_i |T(t)g_i|\|_p &\le \|\tau(0)f - h_0\|_p + \|h_0 - h_t\|_p \\ &< \varepsilon + \varepsilon = 2\varepsilon \quad \text{for} \quad 0 < t < \delta,\end{aligned}$$

and since $S(t)f \ge \tau(t)f \ge \max_i |T(t)g_i|$, it follows that

$$(\tau(0)f - S(t)f)^+ \le (\tau(0)f - \max_i |T(t)g_i|)^+ .$$

This yields

$$\|(\tau(0)f - S(t)f)^+\|_p \le \|(\tau(0)f - \max_i |T(t)g_i|)^+\|_p < 2\varepsilon$$

for $0 < t < \delta$. That is,

$$\lim_{t \to +0} \|(\tau(0)f - S(t)f)^+\|_p = 0. \tag{8}$$

On the other hand, since $T(t)T(0) = T(0)T(t) = T(t)$ implies $\tau(t)\tau(0) \ge \tau(t)$ and $\tau(0)\tau(t) \ge \tau(t)$, it follows that

$$S(t)\tau(0) \ge S(t) \quad \text{and} \quad \tau(0)S(t) \ge S(t) \quad \text{for all} \quad t > 0. \tag{9}$$

Therefore

$$\begin{aligned}(\tau(0)f - S(t)f)^- &\le (\tau(0)f - S(t)\tau(0)f)^- \\ &\le |\tau(0)f - S(t)\tau(0)f|\end{aligned} \tag{10}$$

and

$$(\tau(0)f - S(t)\tau(0)f)^+ \le (\tau(0)f - S(t)f)^+ . \tag{11}$$

By (11) and (8),

$$\lim_{t \to +0} \|(\tau(0)f - S(t)\tau(0)f)^+\|_p \le \lim_{t \to +0} \|(\tau(0)f - S(t)f)^+\|_p = 0.$$

Thus

$$\lim_{t \to +0} \|\tau(0)f - (S(t)\tau(0)f \wedge \tau(0)f)\|_p = 0, \tag{12}$$

whence

$$\lim_{t\to+0}\int (S(t)\tau(0)f \wedge \tau(0)f)^p\,d\mu = \|\tau(0)f\|_p^p. \tag{13}$$

We now use the relations

$$\begin{aligned} 0 &\le \left[S(t)\tau(0)f - (S(t)\tau(0)f\wedge\tau(0)f)\right]^p \\ &\le (S(t)\tau(0)f)^p - (S(t)\tau(0)f\wedge\tau(0)f)^p \quad (\text{because } 1<p<\infty)\end{aligned}$$

and

$$\int (S(t)\tau(0)f)^p\,d\mu \le \|\tau(0)f\|_p^p \quad (\text{because } \|S(t)\|_p \le 1)$$

together with (13) to see that

$$\lim_{t\to+0}\|S(t)\tau(0)f - (S(t)\tau(0)f\wedge\tau(0)f)\|_p = 0. \tag{14}$$

Hence by (12), $\lim_{t\to+0}\|\tau(0)f - S(t)\tau(0)f\|_p = 0$; and (10) gives

$$\lim_{t\to+0}\|(\tau(0)f - S(t)f)^-\|_p \le \lim_{t\to+0}\|\tau(0)f - S(t)\tau(0)f\|_p = 0. \tag{15}$$

This and (8) imply that $\lim_{t\to+0}\|\tau(0)f - S(t)f\|_p = 0$ for all $f \in L_p^+$, completing the proof. □

2. An application

Theorem 2 (cf. Theorem VIII.7.10 in [3] and Theorem 4.3 in [4])**.** *Let $\{T_i(t) : t \ge 0\}$, $i = 1,\dots,d$, be strongly continuous semigroups of linear contractions in L_p, $1 < p < \infty$. Suppose each $T_i(t)$ possesses a majorant $P_i(t)$ such that $\|P_i(t)\|_p \le 1$. Then for every $f \in L_p$ the averages*

$$\begin{aligned} &A(u_1,\dots,u_d)f(x) \\ &\quad = \frac{1}{u_1\dots u_d}\int_0^{u_1}\cdots\int_0^{u_d} T_1(t_1)\dots T_d(t_d)f(x)\,dt_1\dots dt_d \end{aligned} \tag{16}$$

converge a.e. to $T_1(0)\dots T_d(0)f(x)$ as $\max_i u_i \to 0$, and also they converge a.e. to $E_1\dots E_d f(x)$ as $\min_i u_i \to \infty$, where E_i is the operator in L_p defined by

$$E_i f = \lim_{b\to\infty}\frac{1}{b}\int_0^b T_i(t)f\,dt \quad \textit{in } L_p\textit{-norm.}$$

Proof: We first show that the function

$$f^*(x) = \sup_{u_1,\dots,u_d>0}|A(u_1,\dots,u_d)f(x)| \quad (x \in X) \tag{17}$$

is in L_p and satisfies $\|f^*\|_p \le (p/(p-1))^d \|f\|_p$.

For this purpose let $\{S_i(t) : t > 0\}$, $1 \le i \le d$, denote the semigroup moduli of the semigroups $\{T_i(t) : t > 0\}$, $1 \le i \le d$. Write for $u > 0$ and $1 \le i \le d$,

$$A_i(u)f(x) = \frac{1}{u}\int_0^u T_i(t)f(x)\,dt \quad \text{and} \quad B_i(u)|f|(x) = \frac{1}{u}\int_0^u S_i(t)|f|(x)\,dt.$$

Since

$$|A_i(u)f(x)| \le B_i(u)|f|(x) \quad \text{a.e. on } X$$

and

$$\sup_{u>0} B_i(u)|f|(x) = \sup_{u\in Q^+} B_i(u)|f|(x),$$

where Q^+ denotes the set of positive rationals, and for every $u \in Q^+$

$$B_i(u)|f| = \lim_{n\to\infty} \frac{1}{u(n!)} \sum_{m=0}^{u(n!)-1} S_i(m/n!)|f| \quad \text{in } L_p\text{-norm},$$

it follows from the Cantor diagonal argument that there exists a subsequence $\{n'\}$ of the sequence of positive integers such that

$$\sup_{u>0} B_i(u)|f|(x) \le \liminf_{n'\to\infty} f^*_{i,n'}(x) \quad \text{a.e. on } X,$$

where

$$f^*_{i,n}(x) = \sup_{k\ge 1} \frac{1}{k} \sum_{m=0}^{k-1} S_i(m/n!)|f|(x) \quad (n \ge 1).$$

Thus, by Fatou's lemma and Akcoglu's dominated ergodic theorem [1] for positive linear contractions in L_p with $1 < p < \infty$,

$$\|\sup_{u>0} B_i(u)|f|(x)\|_p \le \liminf_{n'\to\infty} \|f^*_{i,n'}\|_p \le \frac{p}{p-1}\|f\|_p\,. \tag{18}$$

Now, the equality $A(u_1,\dots,u_d)f = A_1(u_1)\dots A_d(u_d)f$ implies

$$\begin{aligned} f^*(x) &= \sup_{u_1,\dots,u_d>0} |A_1(u_1)\dots A_d(u_d)f(x)| \\ &\le \sup_{u_1,\dots,u_d>0} B_1(u_1)\dots B_d(u_d)|f|(x) \\ &\le \sup_{u_1,\dots,u_{d-1}>0} B_1(u_1)\dots B_{d-1}(u_{d-1})\ (\sup_{u>0} B_d(u)|f|)(x), \end{aligned}$$

and hence by induction

$$\|f^*\|_p \le \left(\frac{p}{p-1}\right)^{d-1} \|\sup_{u>0} B_d(u)|f|\|_p \le \left(\frac{p}{p-1}\right)^{d} \|f\|_p\,. \tag{19}$$

We apply (19) to infer that the averages $A(u_1,\ldots,u_d)f(x)$ converge a.e. to $T_1(0)\ldots T_d(0)f(x)$ [resp. $E_1\ldots E_d f(x)$] as $\max_i u_i \to 0$ [resp. $\min_i u_i \to \infty$], as follows.

We use an induction argument. Since the set

$$M = \Big\{\frac{1}{b}\int_0^b T_1(t)g(x)\,dt + h : b > 0,\ T_1(0)g = g,\ T_1(0)h = 0\Big\}$$

is dense in L_p, there exists a sequence $\{f_n\}$ in M such that $\lim_n \|f_n - f\|_p = 0$. Since $f_n \in M$ implies

$$\lim_{u\to+0} A_1(u)f_n(x) = T_1(0)f_n(x) \quad \text{a.e. on } X,$$

it follows that the function

$$F(x) = \limsup_{u\to+0} |A_1(u)f(x) - T_1(0)f(x)| \quad (x \in X) \tag{20}$$

satisfies

$$\begin{aligned} F(x) &\le \limsup_{u\to+0} |A_1(u)(f-f_n)(x) - T_1(0)(f-f_n)(x)| \\ &\le \sup_{u>0} |A_1(u)(f-f_n)(x)| + |T_1(0)(f-f_n)(x)|. \end{aligned}$$

Thus

$$\|F\|_p \le \frac{p}{p-1}\|f-f_n\|_p + \|f-f_n\|_p \longrightarrow 0 \quad (n\to\infty).$$

We get $F(x) = 0$ a.e. on X and hence $\lim_{u\to+0} A_1(u)f(x) = T_1(0)f(x)$ a.e. on X.

Next, since L_p is a reflexive Banach space, we see by Eberlein's mean ergodic theorem (cf. [5, Theorem 2.1.5, p. 76]) that there exists a projection operator $E_1 : L_p \to L_p$ for which

$$E_1 f = \lim_{u\to\infty} A_1(u)f \quad \text{in } L_p\text{-norm},$$

and that the set

$$M^\sim = \{g + (h - T_1(s)h) : s > 0,\ T_1(t)g = g \ \text{ for all } \ t > 0\}$$

is dense in L_p. If $g + (h - T_1(s)h) \in M^\sim$, where $T_1(t)g = g$ for all $t > 0$, then

$$\begin{aligned} &A_1(u)[g + (h - T_1(s)h)](x) \\ &\qquad = g(x) + \frac{1}{u}\int_0^s T_1(t)h(x)\,dt - \frac{1}{u}\int_u^{u+s} T_1(t)h(x)\,dt, \end{aligned}$$

and

$$\lim_{u\to\infty}\frac{1}{u}\int_0^s T_1(t)h(x)\,dt=0 \quad \text{a.e. on } X.$$

Letting $n=[u]$ be the integral part of u and k be an integer such that $s<k-1$, we have

$$\begin{aligned}\Big|\frac{1}{u}\int_u^{u+s} T_1(t)h(x)\,dt\Big| &\le \frac{1}{u}\int_u^{u+s} S_1(t)|h|(x)\,dt\\ &\le \frac{1}{n}\int_n^{n+k} S_1(t)|h|(x)\,dt=\frac{1}{n}S_1(n)h^{\sim}(x),\end{aligned}$$

where

$$h^{\sim}(x)=\int_0^k S_1(t)|h|(x)\,dt \quad (x\in X).$$

Define the functions

$$H_n(x)=\sum_{m=n}^{\infty}\Big(\frac{1}{m}S_1(m)h^{\sim}(x)\Big)^p \quad (x\in X). \tag{21}$$

Clearly we get $H_n\ge H_{n+1}\ge\cdots\ge 0$ and

$$\int H_n\,d\mu=\sum_{m=n}^{\infty} m^{-p}\|S_1(m)h^{\sim}\|_p^p\le\Big(\sum_{m=n}^{\infty} m^{-p}\Big)\|h^{\sim}\|_p^p\longrightarrow 0 \quad (n\to\infty).$$

It follows that $\lim_n H_n(x)=0$ a.e. on X, and

$$\lim_{u\to\infty}\Big|\frac{1}{u}\int_u^{u+s} T_1(t)h(x)\,dt\Big|\le\lim_{n\to\infty}\frac{1}{n}S_1(n)h^{\sim}(x)=0$$

a.e. on X. This proves that

$$\lim_{u\to\infty} A_1(u)[g+(h-T_1(s)h)](x)=g(x)=E_1[g+(h-T_1(s)h)](x)$$

a.e. on X. Using this and the density of $M^{\sim}$ in L_p, it follows as before that the function

$$F^{\sim}(x)=\limsup_{u\to\infty}|A_1(u)f(x)-E_1f(x)| \quad (x\in X) \tag{22}$$

satisfies $F^{\sim}=0$ a.e. on X. Thus $\lim_{u\to\infty}A_1(u)f(x)=E_1f(x)$ a.e. on X.

We then use the relation

$$A(u_1,\dots,u_d)f=A(u_1,\dots,u_{d-1})A_d(u_d)f$$

to complete the proof. Since the functions

$$f^{\sim}(u;x) = \sup_{0<r\le u} |A_d(r)f(x) - T_d(0)f(x)| \quad (x \in X) \tag{23}$$

satisfy

$$0 \le f^{\sim}(v;x) \le f^{\sim}(u;x) \in L_p \quad \text{for} \quad 0 < v < u$$

and

$$\lim_{u\to+0} f^{\sim}(u;x) = 0 \quad \text{a.e. on } X,$$

and since

$$\begin{aligned} A(u_1,\dots,u_d)f &- T_1(0)\dots T_d(0)f \\ &= A(u_1,\dots,u_{d-1})[A_d(u_d)f - T_d(0)f] \\ &+ [A(u_1,\dots,u_{d-1}) - T_1(0)\dots T_{d-1}(0)](T_d(0)f), \end{aligned}$$

it follows from the induction hypothesis that the function

$$G(x) = \limsup_{u_1\vee\dots\vee u_d\to 0} |A(u_1,\dots,u_d)f(x) - T_1(0)\dots T_d(0)f(x)| \quad (x \in X) \tag{24}$$

satisfies

$$\begin{aligned} G(x) &\le \limsup_{u_1\vee\dots\vee u_{d-1}\vee u_d\to 0} |A(u_1,\dots,u_{d-1})[A_d(u_d)f - T_d(0)f](x)| \\ &\le \sup_{u_1,\dots,u_{d-1}>0} B_1(u_1)\dots B_{d-1}(u_{d-1})f^{\sim}(u_d;\cdot)(x) \end{aligned}$$

a.e. on X. Hence we get $\|G\|_p \le (\frac{p}{p-1})^{d-1}\|f^{\sim}(u_d;\cdot)\|_p \to 0$ as $u_d \to +0$, by the Lebesgue dominated converge theorem. This implies that $A(u_1,\dots,u_d)f(x) \to T_1(0)\dots T_d(0)f(x)$ a.e. on X as $\max_i u_i \to 0$.

Essentially the same proof can be applied to infer that $A(u_1,\dots,u_d)f(x) \to E_1\dots E_d f(x)$ a.e. on X as $\min_i u_i \to \infty$, and hence we omit the details. □

3. Concluding remarks

(a) In Theorem 1 the hypothesis that $\{T(t) : t > 0\}$ is a contraction semigroup cannot be omitted. In fact, given an $\varepsilon > 0$ there exists a strongly continuous semigroup $\{T(t) : t > 0\}$ of bounded linear operators in L_p, $1 < p < \infty$, such that each $T(t)$ possesses a majorant $P(t)$ satisfying $\|P(t)\|_p < 1+\varepsilon$ and also such that

$$\lim_{m\to\infty} \|(\tau(1/m))^m\|_p = \infty,$$

where $\tau(1/m)$ denotes the linear modulus of $T(1/m)$, $m \ge 1$. An example can be found in [7].

(b) In Theorem 2 the hypothesis that each $T_i(t)$ possesses a majorant $P_i(t)$ such that $\|P_i(t)\|_p \leq 1$ cannot be omitted. In fact, there are negative examples for $p = 2$. More precisely, Akcoglu and Krengel [2] constructed a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of unitary operators in L_2 with $T(0) =$ identity such that the averages $\frac{1}{u}\int_0^u T(t)f(x)\,dt$ diverge a.e. as $u \to +0$ for some f in L_2. Essentially the same idea can be applied to construct another strongly continuous semigroup $\{T(t) : t \geq 0\}$ of unitary operators in L_2 with $T(0) =$ identity such that the averages $\frac{1}{u}\int_0^u T(t)f(x)\,dt$ diverge a.e. as $u \to \infty$ for some f in L_2. See also [5, pp. 191–192].

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