# On the existence of weak solutions for degenerate systems of variational inequalities with critical growth

Martin Fuchs

*Abstract.* We prove the existence of solutions to systems of degenerate variational inequalities.

*Keywords:* variational inequalities, existence

*Classification:* 49

In this note we give a short proof of the following Theorem obtained in [1] not relying on the partial regularity theory.

**Theorem.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and that $p \in (1,\infty)$ is given. For a continuous function $f : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ we consider the variational inequality*

$$\text{(V)} \qquad \begin{cases} \text{find } u \in \mathbb{K} \text{ such that} \\ \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla(v-u)\, dx \ge \int_\Omega f(\cdot, u, \nabla u)\cdot (v-u)\, dx \\ \text{holds for all } v \in \mathbb{K} \end{cases}$$

*where the class $\mathbb{K}$ is defined as $\{v \in H^{1,p}(\Omega,\mathbb{R}^N) : v = u_0 \text{ on } \partial\Omega\,,\ v(x) \in K\}$. Here $K$ denotes the closure of a convex bounded open set in $\mathbb{R}^N$ with the boundary of class $C^2$ and $u_0$ is a given function in $H^{1,p}(\Omega,\mathbb{R}^N)$ such that $u_0(\Omega) \subset K$. Then, if $f$ satisfies the growth estimate*

$$|f(x,y,Q)| \le a \cdot |Q|^p \tag{1}$$

*for some constant $a \ge 0$ and if in addition*

$$a < 1/\operatorname{diam} K \tag{2}$$

*holds, problem* (V) *admits at least one solution $u \in \mathbb{K}$.*

As shown in [1] we obtain as a

**Corollary.** *If* $u_0 \in H^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty$ *is given and if* $f$ *satisfies* (1) *as well as* $a < \frac{1}{2 \cdot \|u_0\|_\infty}$ *, then the Dirichlet problem*

$$\begin{cases} -\partial_\alpha(\,|\nabla u|^{p-2} \partial_\alpha u) = f(\cdot, u, \nabla u) \quad \text{on } \Omega\,, \\ u = u_0 \quad \text{on } \partial\Omega \end{cases}$$

*has at least one weak solution* $u \in H^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty$.

In the quadratic case $p = 2$ the above Theorem is due to Hildebrandt and Widman [5] but we did not succeed to extend their method to general $p$. Our proof (working for all $p$) is based on a compensated compactness type lemma demonstrated in [2] with basic ideas taken from Landes paper [6].

**Lemma.** *Suppose that we have weak convergence* $u_m \rightharpoonup u$ *in the space* $H^{1,p}(\Omega, \mathbb{R}^N)$. *Then there is a subsequence* $\{\tilde{u}_m\}$ *such that* $|\nabla \tilde{u}_m|^{p-2} \nabla \tilde{u}_m \rightharpoonup |\nabla u|^{p-2} \nabla u$ *weakly in* $L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^{nN})$ *and* $\nabla \tilde{u}_m \to \nabla u$ *pointwise a.e. provided we know*

$$\int_\Omega |\nabla u_m|^{p-2}\, \nabla u_m \cdot \nabla \varphi \, dx \le c \cdot \|\varphi\|_\infty$$

*for all* $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$ *with* $0 \le c < \infty$ *independent of* $m$ *and* $\varphi$. □

We now come to the

Proof of the Theorem: For $m \in \mathbb{N}$ let

$$f_m : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N\,,$$

$$f_m(x, y, Q) := \begin{cases} f(x, y, Q): & \text{if } |f(x, y, Q)| \le m \\ \frac{m}{|f(x,y,Q)|} \cdot f(x, y, Q) & \text{else} \end{cases}$$

and consider the approximate problem

$$(\mathrm{V})_m \qquad \begin{cases} \text{find } w \in \mathbb{K} \text{ such that} \\ \int_\Omega |\nabla w|^{p-2}\, \nabla w \cdot \nabla(v - w)\, dx \ge \int_\Omega f_m(\cdot, w, \nabla w) \cdot (v - w)\, dx \\ \text{holds for all } v \in \mathbb{K}\,. \end{cases}$$

As shown in [1] the existence of solutions $u_m$ to $(\mathrm{V})_m$ can be deduced from Schauder's fixed point theorem. Recalling (1), (2) and the definition of $f_m$ we infer

$$(1 - a \cdot \operatorname{diam} K) \cdot \int_\Omega |\nabla u_m|^p \, dx \le \int_\Omega |\nabla u_m|^{p-1} \cdot |\nabla u_0| \cdot dx$$

so that $\sup_m \|u_m\|_{H^{1,p}(\Omega)} < \infty$. Thus we may assume

$$u_m \rightharpoonup u \quad \text{in } H^{1,p}(\Omega, \mathbb{R}^N)$$

at least for a subsequence. In order to proceed further we linearize the variational inequality $(\mathrm{V})_m$ making use of the fact that $\partial K$ is of class $C^2$. As in [3, Theorem 2.1, 2.2] we get for all $\psi \in C_0^1(\Omega, \mathbb{R}^N)$

$$(3) \qquad \begin{cases} \int_\Omega ( |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \psi - f_m(\cdot, u_m, \nabla u_m) \cdot \psi)\, dx \\ = \int_{\Omega \cap [u_m \in \partial K]} \psi \cdot \mathcal{N}(u_m)\, b_m(\cdot, u_m, \nabla u_m)\, dx \end{cases}$$

where $\mathcal{N}(y)$ is the interior normal field of $\partial K$ and $b_m(\cdot, u_m, \nabla u_m)$ has the properties

$$b_m(\cdot, u_m, \nabla u_m) \geq 0 \quad \text{a.e. on } [u_m \in \partial K],$$
$$b_m(\cdot, u_m, \nabla u_m) \leq \tilde{a} \cdot |\nabla u_m|^p$$

with $\tilde{a} \geq 0$ independent of $m$. Now we are in the position to apply the Lemma and deduce

$$(4) \qquad \int_\Omega |\nabla u_m|^{p-2}\, \nabla u_m \cdot \nabla \psi\, dx \to \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla \psi\, dx$$

(after selecting a suitable subsequence). We claim

$$(5) \qquad \int_\Omega f_m(\cdot, u_m, \nabla u_m) \cdot \psi\, dx \to \int_\Omega f(\cdot, u, \nabla u) \cdot \psi\, dx.$$

To prove this we observe

$$\Big(x, u_m(x), \nabla u_m(x)\Big) \to \Big(x, u(x), \nabla u(x)\Big)$$

for almost all $x \in \Omega$, especially

$$f(\cdot, u_m, \nabla u_m) \to f(\cdot, u, \nabla u) \quad \text{a.e.}$$

But for points $x \in \Omega$ with the property that a finite limit $\lim_{m\to\infty} f\big(x, u_m(x), \nabla u_m(x)\big)$ exists, we clearly have

$$f_m\big(x, u_m(x), \nabla u_m(x)\big) = f\big(x, u_m(x), \nabla u_m(x)\big)$$

for $m >> 1$, in conclusion $f_m(\cdot, u_m, \nabla u_m) \to f(\cdot, u, \nabla u)$ a.e. On the other hand the uniform growth estimate $|f_m(x, y, Q)| \leq a \cdot |Q|^p$ combined with the smallness condition (2) implies Caccioppli's inequality

$$\int_{B_{R/2}} |\nabla u_m|^p\, dx \leq \mu \cdot R^{-p} \int_{B_R} |u_m - \big(u_m\big)_R|^p\, dx$$

for any ball $B_R \subset \Omega$ with $\mu$ independent of $m$. From this we easily get

$$\sup_m \|\nabla u_m\|_{L^q(\Omega')} < \infty$$

for any subregion $\Omega' \subset\subset \Omega$ and with $q$ slightly larger as $p$. After passing to a subsequence we may therefore assume

$$f_m(\cdot, u_m, \nabla u_m) \rightharpoonup: g$$

weakly in the space $L^{q/p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for some function $g$. Using Egoroff's Theorem we find $g = f(\cdot, u, \nabla u)$ which proves (5).

Next we look at the remaining integral

$$\int_{[u_m \in \partial K]} \psi \cdot \mathcal{N}(u_m) \cdot b_m(\cdot, u_m, \nabla u_m)\, dx := I_m$$

and specialize $\psi = v - u$ where $v \in \mathbb{K}$ is arbitrary but with the property spt $(v - u) \subset\subset \Omega$. (Note that (4), (5) remain valid). We have

$$\begin{aligned} I_m = \quad & \int_{[u_m \in \partial K] \cap \operatorname{spt}(v-u)} (v - u_m) \cdot \mathcal{N}(u_m) \cdot b_m(\cdot, u_m, \nabla u_m)\, dx \\ & + \int_{[u_m \in \partial K] \cap \operatorname{spt}(v-u)} (u_m - u) \cdot \mathcal{N}(u_m) \cdot b_m(\cdot, u_m, \nabla u_m)\, dx \\ =: \quad & I_m^1 + I_m^2 , \end{aligned}$$

$I_m^1 \geq 0$ an account of $(v - u_m) \cdot \mathcal{N}(u_m) \geq 0$ a.e. on $[u_m \in \partial K] \cap \operatorname{spt}(v - u)$ (due to the convexity of $K$) and

$$\begin{aligned} |I_m^2| & \leq \int_{\operatorname{spt}(u-v)} \tilde{a} \cdot |\nabla u_m|^p\, |u_m - u|\, dx \\ & \leq \tilde{a} \cdot \Big( \int_{\operatorname{spt}(u-v)} |\nabla u_m|^q\, dx \Big)^{p/q} \cdot \Big( \int_{\operatorname{spt}(u-v)} |u_m - u|^{\frac{q}{q-p}}\, dx \Big)^{1-p/q} \\ & \xrightarrow[m \to \infty]{} 0, \end{aligned}$$

since $\|\nabla u_m\|_{L^q\left(\operatorname{spt}(u-v)\right)}$ is uniformly bounded and

$$\int_{\operatorname{spt}(u-v)} |u_m - u|^{\frac{q}{q-p}}\, dx \leq \mathrm{const}(q, p, \operatorname{diam} K) \cdot \int_{\operatorname{spt}(u-v)} |u_m - u|^p\, dx \longrightarrow 0.$$

Putting together our results we arrive at

$$\text{(6)} \qquad \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla(v - u)\, dx \geq \int_\Omega f(\cdot, u, \nabla u) \cdot (v - u)\, dx$$

for all $v \in \mathbb{K}$ such that $\operatorname{spt}(u-v) \subset\subset \Omega$. We have to remove the support condition on $v \in \mathbb{K}$. To this purpose consider an arbitrary function $v \in \mathbb{K}$. Then $v - u \in \overset{\circ}{H}{}^{1,p}(\Omega, \mathbb{R}^N)$ so that there is a sequence $w_m \in C_0^\infty(\Omega, \mathbb{R}^N)$ such that $w_n \to v - u$ in the strong topology of the space $H^{1,p}(\Omega, \mathbb{R}^N)$. Let $F : \mathbb{R}^N \to K$ denote the projection onto the set $K$. Then $v_m := F(u + w_m)$ belongs to the class $\mathbb{K}$, moreover (6) is valid for $v_m$. It is easy to check that

$$v_m \rightharpoonup F(v) = v$$

weakly in $H^{1,p}(\Omega, \mathbb{R}^N)$, hence

$$\int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla(v_m - u)\, dx \to \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla(v - u)\, dx.$$

After passing to a subsequence we may assume $v_m \to v$ a.e. on $\Omega$ and since

$$|f(\cdot, u, \nabla u)| \cdot |v_m - u| \le a \cdot \operatorname{diam} K \cdot |\nabla u|^p \in L^1(\Omega)$$

we deduce from dominated convergence that

$$\int_\Omega f(\cdot, u, \nabla u) \cdot (v_m - u) dx \to \int_\Omega f(\cdot, u, \nabla u) \cdot (v - u)\, dx$$

so that $u$ is a solution of the variational inequality (V). □

From [4] we get in addition

**Corollary.** *Let $u$ denote the solution of* (V) *obtained in the Theorem. Then there is a relatively closed set $\Sigma \subset \Omega$ such that $u \in C^{1,\alpha}(\Omega - \Sigma)$ for some $0 < \alpha < 1$ and $\mathcal{H}^{n-p}(\Sigma) = 0$.*

## References


[1] Fuchs M, *Existence via partial regularity for degenerate systems of variational inequalities with natural growth*, Comment. Math. Univ. Carolinae **33** (1992), 427–435.

[2] ———, *The blow up of p-harmonic maps*, Manus. Math. **81** (1993), 89–94.

[3] ———, *p-harmonic obstacle problems*, Annali di Mat. Pura Appl. **156** (1990), 127–180.

[4] ———, *Smoothness for systems of degenerate variational inequalities with natural growth*, Comment. Math. Univ. Carolinae **33** (1992), 33–41.

[5] Hildebrandt S., Widman K.-O., *Variational inequalities for vector-valued functions*, J. Reine Angew. Math. **309** (1979), 181–220.

[6] Landes R., *On the existence of weak solutions of perturbed systems with critical growth*, J. Reine Angew. Math. **393** (1989), 21–38.


Universität des Saarlandes, Fachbereich Mathematik, D-6600 Saarbrücken, Germany

(*Received* July 29, 1993)