Large cardinals and Dowker products

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Abstract. We prove that if there is a model of set-theory which contains no first countable, locally compact, scattered, countably paracompact space X, whose Tychonoff square is a Dowker space, then there is an inner model which contains a measurable cardinal.

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In this paper we always take space to mean Hausdorff topological space. A space is normal if every pair of disjoint closed sets can be separated by disjoint open sets, and binormal if its product with the closed unit interval is normal. A space is countably paracompact (metacompact) if every countable open cover has a locally (point) finite open refinement. In [Dk], Dowker shows that a normal space is binormal iff it is countably paracompact iff it is countably metacompact. A Dowker space is a normal space that is not countably paracompact. For a survey of Dowker spaces we refer the reader to [R].

Rudin and Starbird [RS] have shown that for normal, countably paracompact X and metrizable M, $X \times M$ is normal if and only if it is countably paracompact. They asked whether a product of two normal, countably paracompact spaces could be a Dowker space. Bešlagić constructs various positive answers to this question, assuming $\diamondsuit$ or CH, in [B1], [B2] & [B3].

In [G] we prove that if there is a model of set theory which contains no first countable, locally compact, scattered Dowker spaces, then there is a model of set-theory which contains a measurable cardinal. Here we extend this result by proving that large cardinals are needed for a model in which there is no first countable, locally compact, countably paracompact space X with first countable, locally compact, scattered Dowker square:

1. Theorem. *If no inner model of set theory contains a measurable cardinal, then there is a first countable, locally countable, locally compact, strongly zero-dimensional, collectionwise normal, countably paracompact, scattered space whose Tychonoff square is a first countable, locally compact, collectionwise normal, scattered Dowker space.*

Notation and terminology are standard—see [E], [K] or [KV]. We regard cardinals as initial ordinals, and an ordinal as the set of its predecessors. We use the term *club set* or *club* to denote a closed, unbounded subset of an ordinal—it will be clear from the context which particular ordinal we mean. For a function $f : A \to B$, we denote by $f``C$ the set $\{f(x) : x \in C \subseteq A\}$. For a subset A of $\alpha \times \beta$, we denote the set $\{\gamma : (\exists\delta)(\gamma, \delta) \in A\}$ by $\operatorname{dom} A$, and the set $\{\delta : (\exists\gamma)(\gamma, \delta) \in A\}$ by $\operatorname{ran} A$. Following [B1], a subset A of $\kappa^+ \times \kappa^+$ is said to be 2-unbounded if A is not a subset of $(\kappa^+ \times \alpha) \cup (\alpha \times \kappa^+)$ for any $\alpha \in \kappa^+$. As usual we use the following characterization from [Dk]: a space is countably metacompact if and only if, for every decreasing sequence $\{D_n\}_{n\in\omega}$ of closed subsets of X, which has empty intersection, there is a sequence $\{U_n\}_{n\in\omega}$ of open sets, U_n containing D_n for each n, which also has empty intersection.

A stationary subset E of some uncountable cardinal λ is said to be *non-reflecting* if, for every $\alpha < \lambda$, $\alpha \cap E$ is non-stationary in α. If E is a non-reflecting stationary subset of κ^+ and $\alpha \in \kappa^+$, then it is easy to see that there is a club set $H = \{\gamma_\lambda : \lambda \in \theta \leq \alpha\}$ of α such that H and E are disjoint, and $(\gamma_\lambda, \gamma_{\lambda+1})$ is countable for all $\lambda \in \theta$. In what follows we shall let E denote a non-reflecting, stationary subset of κ^+, each member of which has countable cofinality.

2. Definition. $\clubsuit_{\kappa^+}(E, 2)$ is the assertion that there is a collection $\{R_{\alpha,i} : R_{\alpha,i} \subseteq \alpha,\ \alpha \in E \cap \textsc{lim} \text{ and } i \in 2\}$ such that each $R_{\alpha,i}$ is an ω-sequence, cofinal in α, and $\{\alpha \in E : R_{\alpha,i} \subseteq X_i \text{ for both } i \in 2\}$ is stationary whenever X_0 and X_1 are unbounded subsets of κ^+. □

In [G] we deduce, via [Dv], [DJ] and [F],

3. Lemma. *If no inner model of set-theory contains a measurable cardinal, then* $\clubsuit_{\kappa^+}(E, 2)$ *for some* κ^+. □

In the construction of the space X, we use the following two consequences of $\clubsuit_{\kappa^+}(E, 2)$.

4. Definition. $\clubsuit_{\kappa^+\times\kappa^+}(E, 2)$ is the assertion that there is a sequence $\{S_{\alpha,i} : S_{\alpha,i} \subseteq \alpha \times \alpha,\ \alpha \in E \cap \textsc{lim} \text{ and } i \in 2\}$ such that $S_{\alpha,i}$ is an ω-sequence, cofinal in $\alpha \times \alpha$, and $\{\alpha \in E : S_{\alpha,i} \subseteq X_i\ i \in 2\}$ is stationary whenever X_0 and X_1 are 2-unbounded subsets of $\kappa^+ \times \kappa^+$. □

5. Definition. $\clubsuit^{\cap}_{\kappa^+}(E, 2)$ is the assertion that there is a sequence $\{T_{\alpha,i,n} : T_{\alpha,i,n} \subseteq \alpha,\ \alpha \in E \cap \textsc{lim}, \text{ and } i \in 2\}$ such that $\bigcup_{n\in\omega} T_{\alpha,i,n}$ and each $T_{\alpha,i,n}$ is an ω-sequence, cofinal in α, $T_{\alpha,i,n} \cap T_{\alpha,j,m}$ is empty whenever $i \neq j$ or $m \neq n$, and $\bigcap_{n\in\omega}\{\alpha \in E : T_{\alpha,i,n} \subseteq X_{i,n}, \text{ for both } i \in 2\}$ is stationary whenever $\{X_{i,n} : i \in 2,\ n \in \omega\}$ is a collection of unbounded subsets of κ^+. □

6. Lemma. *If* $\clubsuit_{\kappa^+}(E, 2)$, *then* $\clubsuit_{\kappa^+\times\kappa^+}(E, 2)$ *and* $\clubsuit^{\cap}_{\kappa^+}(E, 2)$.

Proof: Let $\{R_{\alpha,i} : \alpha \in E \cap \textsc{lim},\ i \in 2\}$ be a $\clubsuit_{\kappa^+}(E, 2)$-sequence. We may assume that $R_{\alpha,0}$ and $R_{\alpha,1}$ are disjoint for all α in $E \cap \textsc{lim}$. Let $f : \kappa^+ \to \kappa^+ \times \kappa^+$ and $g : \kappa^+ \to \kappa^+ \times \omega$ be any bijections. $F = \{\alpha : f``\alpha = \alpha \times \alpha\}$ and $G = \{\alpha : g``\alpha = \alpha \times \omega\}$ are both club in κ^+.

For α in $E \cap F \cap \text{LIM}$ such that both $f``R_{\alpha,0}$ and $f``R_{\alpha,1}$ are cofinal in $\alpha \times \alpha$, define $S_{\alpha,i}$ to be the set $f``R_{\alpha,i}$. Otherwise, for α in E, let $S_{\alpha,i}$ be an arbitrary sequence cofinal in $\alpha \times \alpha$. It is easy to see that $\{S_{\alpha,i} : S_{\alpha,i} \subseteq \alpha \times \alpha,\ \alpha \in E \cap \text{LIM}$ and $i \in 2\}$ is a $\clubsuit_{\kappa^+ \times \kappa^+}(E,2)$-sequence.

If α is in $E \cap G \cap \text{LIM}$, $i \in 2$ and $n \in \omega$, let $T_{\alpha,i,n}$ be the set $\operatorname{dom}(g``B_{\alpha,i} \cap (\alpha \times \{n\}))$. Otherwise, for α in E, let $T_{\alpha,i,n}$ be arbitrary.

To see that $\{T_{\alpha,i,n} : T_{\alpha,i,n} \subseteq \alpha,\ \alpha \in E \cap \text{LIM}$, and $i \in 2\}$ is a $\clubsuit^{\cap}_{\kappa^+}(E,2)$-sequence, let $\{X_{i,n}\}_{\substack{i\in 2\\ n\in\omega}}$ be a collection of unbounded subsets of κ^+, and let $X_i = \bigcup_{n\in\omega} X_{i,n} \times \{n\}$. $S = \{\alpha \in E : R_{\alpha,i} \subseteq g^{-1}``X_i,\ i \in 2\}$ is stationary. If $\alpha$ is in $S$, then $g``R_{\alpha,i}$ is a subset of X_i and hence S is a subset of $\bigcap\{\alpha \in E : T_{\alpha,i,n} \subseteq X_{i,n},\ i \in 2\}$. $\square$

Our construction is similar to that used by Bešlagić in [B1]. We define three normal topologies, $\mathcal{T}_i$, $i \in 3$, on the point set $Y = \kappa^+ \times \omega$. The topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ both refine $\mathcal{T}_2$, which is a Hausdorff topology, hence the diagonal Δ of $(Y,\mathcal{T}_0) \times (Y,\mathcal{T}_1)$ is a closed subspace of X^2. Our space X is the disjoint topological sum of $(Y,\mathcal{T}_0)$ and $(Y,\mathcal{T}_1)$. $\clubsuit_{\kappa^+ \times \kappa^+}(E,2)$ helps to ensure that the product X^2 is normal, and that Δ is a Dowker space. Since Δ is closed in X^2, X^2 is also a Dowker space. We use $\clubsuit^{\cap}_{\kappa^+}(E,2)$ to ensure that $(Y,\mathcal{T}_i)$, $i \in 2$ is countably paracompact (cf § 5 [B1]).

7. Example. $\clubsuit_{\kappa^+}(E,2)$ *There is a first countable, locally countable, locally compact, strongly zero-dimensional, collectionwise normal, countably paracompact, scattered space X, whose Tychonoff square is a first countable locally compact, collectionwise normal, scattered Dowker space.*

Proof: Let Y be the point set $\kappa^+ \times \omega$, let $\pi : Y \to \kappa^+$ be the natural projection, $\pi((\alpha,n)) = \alpha$, and let $\Pi : Y^2 \to {\kappa^+}^2$ be the natural projection, $\Pi((\alpha,n),(\beta,m)) = (\alpha,\beta)$. Let $\{S_{\alpha,i} : S_{\alpha,i} \subseteq \alpha \times \alpha,\ \alpha \in E \cap \text{LIM}$ and $i \in 2\}$ and $\{T_{\alpha,i,n} : T_{\alpha,i,n} \subseteq \alpha,\ \alpha \in E \cap \text{LIM},\ n \in \omega$ and $i \in 2\}$ be $\clubsuit_{\kappa^+ \times \kappa^+}(E,2)$- and $\clubsuit^{\cap}_{\kappa^+}(E,2)$-sequences respectively. Bearing in mind the proof of Lemma 6, it is not hard to see that we may assume that $\bigcup_{i\in 2}(\operatorname{ran} S_{\alpha,i} \cup \operatorname{dom} S_{\alpha,i})$ and $\bigcup_{\substack{i\in 2\\ n\in\omega}} T_{\alpha,i,n}$ are disjoint for each α in $E \cap \text{LIM}$. We may also assume that each ω-sequence $S_{\alpha,i}$ is strictly increasing in both coordinates.

For each α in $E \cap \text{LIM}$ and each $i \in 2$, partition $S_{\alpha,i}$ into ω disjoint sequences $S_{\alpha,i,n}$, where $n \in \omega$, each cofinal in $\alpha \times \alpha$. Let $B(\alpha,i,n)$ be the sequence $\operatorname{dom} S_{\alpha,i,n} \cup \operatorname{ran} S_{\alpha,i,n}$. For each $n \in \omega$, $B(\alpha,i,n)$ is an ω-sequence, cofinal in α, and the collection $\{B(\alpha,i,n) : \alpha \in E \cap \text{LIM},\ i \in 2\}$ is a $\clubsuit_{\kappa^+}(E,2)$-sequence. Since $S_{\alpha,i}$ is strictly increasing in both coordinates, $B(\alpha,i,n)$ and $B(\alpha,i,m)$ are disjoint whenever $n \neq m$. Let $B(\alpha,n) = B(\alpha,0,n) \cup B(\alpha,1,n)$ and let $B(\alpha) = \bigcup_{n\in\omega} B(\alpha,n)$. Enumerate the ω-sequence $B(\alpha)$ increasingly as $\{\beta(\alpha,j) : j \in \omega\}$.

For $i \in 2$ let $C(\alpha,i,n) = T_{\alpha,i,n}$ and let $C(\alpha,2,n) = C(\alpha,0,n) \cup C(\alpha,1,n)$. Let $C(\alpha) = \bigcup_{n\in\omega} C(\alpha,2,n)$. Enumerate the ω-sequence $C(\alpha)$ increasingly as $\{\gamma(\alpha,j) : j \in \omega\}$.

By assumption $B(\alpha)$ and $C(\alpha)$ are disjoint for all α in $E \cap \text{LIM}$. Let $A(\alpha) = B(\alpha) \cup C(\alpha)$ and index $A(\alpha)$ increasingly as $\{\alpha(k) : k \in \omega\}$.

We define the topologies $\mathcal{T}_i$ by induction on the lexicographical order on $\kappa^+ \times \omega$. At each stage of the induction (α, n), and for each $i \in 3$, we define a topology $\mathcal{T}_{i,\alpha}$ on $Y_\alpha = \alpha \times \omega$ and then a neighbourhood base $\mathcal{N}_i(\alpha, n) = \{N_i((\alpha, n), k)\}_{k \in \omega}$ at the point (α, n). Our inductive hypotheses are, for $\gamma < \beta < \alpha$ and $i \in 3$:

(1) $\mathcal{T}_{i,\beta}$ is a Hausdorff, conservative extension of $\mathcal{T}_{i,\gamma}$, and $Y_{\gamma+1}$ is a $\mathcal{T}_{i,\beta}$-clopen subset of Y_β;
(2) $\mathcal{N}_i(\gamma, k)$ is a decreasing neighbourhood base of sets which are clopen, compact and countable under $\mathcal{T}_{i,\beta}$, and are subsets of $Y_{\gamma+1}$;
(3) $N_i((\beta, n), k)$ and $N_i((\beta, m), k)$ are disjoint whenever $n \neq m$;
(4) $N_0((\beta, n), k) \cup N_1((\beta, n), k)$ is a subset of $N_2((\beta, n), k)$ for all $k \in \omega$;
(5) if $\delta(n, k) = \inf\{\pi``N_i((\alpha, n), k)\}$, then, for all $n \in \omega$, the sequence $\{\delta(n, k) : k \in \omega\}$ is cofinal in α;
(6) for all $0 < r \in \omega$, the point $(\gamma, 0)$ is a $\mathcal{T}_{0,\beta}$-limit of each sequence $C(\gamma, 0, r) \times \{r\}$, a $\mathcal{T}_{1,\beta}$-limit of each sequence $C(\gamma, 1, r) \times \{r\}$, and a $\mathcal{T}_{2,\beta}$-limit of both sequences;
(7) if $N_0 \in \mathcal{N}_0(\beta, 0)$ and $N_1 \in \mathcal{N}_1(\beta, 0)$, then $N_0 \cap N_1 = \{(\beta, 0)\}$;
(8) for all $0 \leq p \leq m$, the point $(\beta, m+1)$ is a $\mathcal{T}_i$-limit of the sequence $B(\alpha, m) \times \{p\}$.

If $\alpha = 0$, let $\mathcal{T}_{i,0} = \varnothing$ and let $\mathcal{N}_i(0, n) = \{\{(0, n)\}\}$ for each $i \in 3$. Suppose that we have defined $\mathcal{N}_i(\beta, k)$ for each $i \in 3$, all $\beta \in \alpha$ and all $k \in \omega$. Define $\mathcal{T}_{i,\alpha}$ to be the topology generated by $\bigcup\{\mathcal{N}_i(\beta, k) : k \in \omega,\ \beta < \alpha\}$.

If $\alpha = \beta + 1$ for some β, or α is not in E, then we declare the point (α, n) to be isolated and define $\mathcal{N}_i(\alpha, n)$ to be $\{\{(\alpha, n)\}\}$ for each $i \in 3$.

Now suppose that α is a limit ordinal in E.

First let us suppose that $n = 0$. The sequence $C(\alpha)$ is enumerated as $\{\gamma(\alpha, j) : j \in \omega\}$. Each $\gamma(\alpha, j)$ in $C(\alpha)$ occurs uniquely in T_{α, i_j, r_j} for some $i_j \in 2$ and some $r_j \in \omega$, and is indexed as $\alpha(k_j)$ in $A(\alpha)$. By inductive hypotheses (4) and (5), whenever $r_j > 0$, we can choose a basic open set $N_2(\gamma(\alpha, j), r_j)$ from $\mathcal{N}_2(\gamma(\alpha, j), r_j)$ such that

(†) $\pi``N_2(\gamma(\alpha, j), r_j)$ is a subset of the interval $(\alpha(k_j - 1), \alpha(k_j)]$ in κ^+ (by (5)).

For $i \in 3$, and each $k \in \omega$, define
$$N_i((\alpha, 0), k) = \{(\alpha, 0)\} \cup \bigcup\{N_2(\gamma(\alpha, j), r_j) : \gamma(\alpha, j) \in C(\alpha, i, r_j),\ r_j > 0,\ j > k\}.$$

Now suppose that $n = m+1$ for some $m \in \omega$. The sequence $B(\alpha)$ is enumerated as $\{\beta(\alpha, j) : j \in \omega\}$, and each $\beta(\alpha, j)$ occurs uniquely in some $B(\alpha, r_j)$, and is indexed in $A(\alpha)$ as $\alpha(k_j)$. By (4), (5) and the fact that $\mathcal{T}_{2,\alpha}$ is Hausdorff, for each $\beta(\alpha, r_j)$ such that $r_j = m$ and for each $p \leq r_j$, we can choose disjoint basic open neighbourhoods $N_i(\beta(\alpha, j), p)$ from $\mathcal{N}_i(\beta(\alpha, j), p)$, $i \in 3$ of the point $(\beta(\alpha, j), p)$ such that $N_i(\beta(\alpha, j), p)$ is a subset of $N_2(\beta(\alpha, j), p)$, for each $i \in 2$, and

(‡) $\pi``N_i(\beta(\alpha, j), p)$ is a subset of the interval $(\alpha(k_j - 1), \alpha(k_j)]$ in κ^+.

For $i \in 3$, and each $k \in \omega$, define

$$N_i((\alpha,n),k) = \\ = \{(\alpha,n)\} \cup \bigcup\{N_i(\beta(\alpha,j),p) : \beta(\alpha,j) \in B(\alpha,m),\ p \leq m, \text{ and } j > k\}.$$

It is not hard to check that the inductive hypotheses hold.

Let $\mathcal{T}_i$ be the topology generated by $\bigcup_{(\alpha,n)\in Y} \mathcal{N}_i(\alpha,n)$.

Clearly both $\mathcal{T}_0$ and $\mathcal{T}_1$ refine $\mathcal{T}_2$, and it is not hard to check that each $(Y,\mathcal{T}_i)$ is Hausdorff. Moreover, in each of these topologies, a point (α,n) of Y is either isolated or has a neighbourhood homeomorphic to the ordinal space $\omega^m + 1$, for some $m \leq n$. Therefore, for each $i \in 3$, $(Y,\mathcal{T}_i)$ is regular, first countable, locally countable, locally compact, zero-dimensional and locally metrizable.

Claim 1. *For each $i \in 3$ and all $\alpha \in \kappa^+$, the subspace $Y_\alpha = \alpha \times \omega$ of $(Y,\mathcal{T}_i)$ is metrizable.*

Proof of Claim 1: Fix $i \in 3$. The proof is by induction, so assume that Y_β is metrizable for all $\beta \in \alpha$.

Since E is a non-reflecting stationary set, each of whose elements has countable cofinality, if α is a limit ordinal (either in E or not), or $\alpha \leq \omega_1$, then there is a sequence $\{\alpha_\gamma : \gamma \in \theta \leq \alpha\}$, which is both closed, cofinal in α, and disjoint from E. But then

$$\{(\alpha_\gamma,\alpha_{\gamma+1}) \times \omega : \gamma \in \theta\} \cup \bigcup\{\{\alpha_\gamma\} \times \omega : \gamma \in \theta\}$$

partitions Y_α into disjoint, clopen, metrizable subsets.

Now suppose that $\alpha = \beta + 1$. Without loss of generality, we may assume that β is a limit ordinal. If β is not in E, then the two sets Y_β and $\{(\beta,n) : n \in \omega\}$ partition Y_α into disjoint, clopen, metrizable sets, and we are done. Assume that β is an element of E. By construction, $\{N_j\}_{j\in\omega}$, where $N_j = N_i((\beta,j),1) \in \mathcal{N}_i(\beta,j)$, forms a disjoint collection of clopen, metrizable subsets of Y_α. Furthermore, by † and ‡, if x_j is any point of N_j, then the set $\{\pi(x_j)\}_{j\in\omega}$ forms an ω-sequence, cofinal in β (though not necessarily indexed in increasing order), so the only possible limit point of the sequence $\{x_j\}_{j\in\omega}$ is (β,k) for some k in ω, which is impossible. Therefore $\{N_j\}_{j\in\omega}$ is a discrete collection of countable, clopen sets. But now $N = \bigcup_{j\in\omega} N_j$ and $Z = Y_\alpha - N$ partition Y_α into disjoint, clopen, metrizable subspaces, and again Y_α is metrizable. □

Claim 2. *Fix $i \in 3$. If H is a subset of $(Y,\mathcal{T}_i)$ of size κ^+, then H has a limit point, and, if C and D are closed subsets of $(Y,\mathcal{T}_i)$, both of size κ^+, then C and D are not disjoint.*

Proof of Claim 2: For any subset A of Y let $A(n) = A \cap (\kappa^+ \times \{n\})$.

Suppose that H has size κ^+, then $H(n)$ also has size κ^+ for some $n \in \omega$. By $\clubsuit_{\kappa^+\times\kappa^+}(E,2)$, $B(\alpha,n) \times \{n\}$ is contained in $H(n)$, for some α in E, so H has

$(\alpha, n+1)$ as a limit point in $\kappa^+ \times \{n+1\}$. In fact, since κ^+ is a regular cardinal, H has κ^+ limit points in $\kappa^+ \times \{n+1\}$.

Now let C and D be closed subsets of $(Y, \mathcal{T}_i)$ of cardinality κ^+. From the previous paragraph it is clear that $|C(n)| = |D(n)| = \kappa^+$, for some n. By $\clubsuit_{\kappa^+ \times \kappa^+}(E, 2)$, there is an α in E for which both $B(\alpha, 0, n) \times \{n\}$ is a subset of $C(n)$, and $B(\alpha, 1, n) \times \{n\}$ is a subset of $D(n)$, so C and D have a common limit point. □

For each $i \in 3$, the (strong) collectionwise normality of $(Y, \mathcal{T}_i)$ is immediate from Claims 1 and 2: Let $\mathcal{D}$ be a discrete collection of closed sets. By Claim 2, $\mathcal{D}$ has size less than κ^+ and there is some successor α such that Y_α contains all but at most one of the sets in $\mathcal{D}$. Since Y_α is clopen and metrizable we are done.

The strong zero-dimensionality of $(Y, \mathcal{T}_i)$ also follows from Claims 1 and 2: Suppose that A and B are subsets of Y which are completely separated by the function $f : (Y, \mathcal{T}_i) \to [0, 1]$ in that $f``A = \{0\}$ and $f``B = \{1\}$. The sets $f^{-1}``[0, 1/4]$ and $f^{-1}``[3/4, 1]$ are disjoint, closed sets containing A and B respectively, so, as above, there is a successor α such that Y_α contains A, say. Y_α is a metrizable, locally compact, zero-dimensional subspace of Y and is, therefore, strongly zero-dimensional (by 6.2.10 [E]).

Claim 3. *$(Y, \mathcal{T}_i)$ is countably paracompact for each $i \in 3$.*

Proof of Claim 3: Fix $i \in 3$. Since $(Y, \mathcal{T}_i)$ is normal it suffices to show that, for every decreasing sequence of closed subsets $\{D_n\}_{n \in \omega}$ of $(Y, \mathcal{T}_i)$ with empty intersection, there is a sequence of open subsets $\{U_n\}_{n \in \omega}$ with empty intersection such that U_n contains D_n.

Let $\{D_n\}_{n \in \omega}$ be such a sequence of closed sets. Suppose that each D_n has size κ^+, then, with the notation used above, Claim 2 implies that $D_n(k)$ has size κ^+ for all k greater than some $k_n \in \omega$. By relabelling and adding repetitions if necessary, we may assume that $D_n(n)$ has size κ^+ for all n larger than some $n_0 > 0$. Now, by $\clubsuit^{\cap}_{\kappa^+}(E, 2)$,

$$S = \bigcap_{n \in \omega} \{\alpha \in E : T_{\alpha, i, n} \subseteq D_n(n),\ i \in 2\}$$

is a stationary set, and therefore non-empty. By the construction of the topology $\mathcal{T}_i$, if α is in S, then $(\alpha, 0)$ is in D_n for all $n \in \omega$, and so $\bigcap D_n$ is not empty—a contradiction.

Pick n_0 such that $|D_n| \leq \kappa$ for all $n \geq n_0$. By Claim 1 there is a successor α such that D_n is a subset of Y_α for $n \geq n_0$. The claim follows since Y_α is clopen and metrizable. We are done. □

Claim 4. *For $i, j \in 2$, $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$ is normal.*

Proof of Claim 4: Let C and D be disjoint closed subsets of $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$, and recall that $\Pi : (\kappa^+ \times \omega)^2 \to \kappa^+ \times \kappa^+$ is the natural projection.

Suppose that both $\Pi``C$ and $\Pi``D$ are 2-unbounded in $\kappa^+ \times \kappa^+$. There are integers $m, n, j, k \in \omega$ such that $C_{n,k} = \{(\gamma, \delta) : ((\gamma, n), (\delta, k)) \in C\}$ and $D_{m,j} = \{(\gamma, \delta) : ((\gamma, m), (\delta, j)) \in D\}$ are both 2-unbounded. Let $s = n + m + j + k + 1$, so that s is strictly greater than n, m, j and k. By $\clubsuit_{\kappa^+ \times \kappa^+}(E, 2)$, there is some α in E such that $S_{\alpha,0}$ is a subset of $C_{n,k}$ and $S_{\alpha,1}$ is a subset of $D_{m,j}$. By the definition of the sequence $B(\alpha, 0, s)$

$$C_{n,k} \cap (B_{\alpha,0,s} \times B(\alpha, 0, s))$$

is infinite and cofinal in (α, α). By the definition of the topologies $\mathcal{T}_i$ and $\mathcal{T}_j$,

$$C \cap \big((B(\alpha, 0, s) \times \{n\}) \times (B_{\alpha,0,s} \times \{k\})\big)$$

is cofinal in $((\alpha, s), (\alpha, s))$, which is therefore a limit point of C. Similarly $((\alpha, s), (\alpha, s))$ is a limit point of D, and C and D are not disjoint.

So suppose that $\Pi``C$ is not 2-unbounded. Choose γ not in E such that C is a subset of

$$K = ((\gamma \times \omega) \times (\kappa^+ \times \omega)) \cup ((\kappa^+ \times \omega) \times (\gamma \times \omega)).$$

Since γ is not in E, K is a clopen subset of $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$. Since E is a non-reflecting stationary set, there is a club set H of γ, enumerated as $\{\gamma_\lambda : \lambda \in \theta \leq \gamma\}$, which misses E and such that $G_\lambda = \{\alpha : \gamma_\lambda < \alpha < \gamma_{\lambda+1}\}$ is countable. Now $\{\{\gamma_\lambda\} \times \omega\}_{\lambda \in \theta} \cup \{G_\lambda \times \omega\}_{\lambda \in \theta}$ partitions Y_γ into countable, metrizable, $\mathcal{T}_i$-clopen subsets of Y, for $i = 0$ or 1. Lemma 2.8 of [B1] states that, for normal, countably paracompact space X and a countable metric space M, $X \times M$ is normal. It is easy to see, then, that K is normal. Since K is clopen, $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$ is now, itself, seen to be normal—proving the claim. □

The proof that $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$ is collectionwise normal is similar.

Now, let X be the disjoint topological sum of $(Y, \mathcal{T}_0)$ and $(Y, \mathcal{T}_1)$. From the above, it is clear that X satisfies the properties listed in the statement of the Theorem 1, except that it remains to show that X^2 is not countably paracompact:

Claim 5. *The closed subspace* $\Delta = \{((\alpha, n), (\alpha, n)) : \alpha \in \kappa^+, n \in \omega\}$ *of* $(Y, \mathcal{T}_0) \times (Y, \mathcal{T}_1)$ *is not countably metacompact.*

Proof of Claim 5: Let $D_n = \{((\alpha, j), (\alpha, j)) : \alpha \in \kappa^+, j \geq n\}$, and let U_n be any open subset of Δ containing D_n. $\{D_n\}_{n \in \omega}$ is a decreasing sequence of closed subsets of Δ with empty intersection, so it is enough to show that $\bigcap U_n$ is non-empty.

Notice that, since the sequences $C(\alpha, 0)$ and $C(\alpha, 1)$ are disjoint, the point $((\alpha, 0), (\alpha, 0))$ is isolated for each $\alpha \in \kappa^+$ (by hypothesis (7)). However, if α is a limit in E, then $(\alpha, n + 1)$ is both a $\mathcal{T}_0$- and a $\mathcal{T}_1$-limit of the sequence $B(\alpha, n) \times \{n\}$. So, as $\{B(\alpha, i, n)\}_{\substack{i \in 2 \\ \alpha \in E \cap \mathrm{LIM}}}$ is a $\clubsuit_{\kappa^+}(E, 2)$-sequence, the proof of Claim 2 is, almost verbatim, a proof of:

* If H is a subset of Δ of size κ^+, then H has a limit point in Δ, and, if C and D are closed subsets of Δ, both of size κ^+, then C and D are not disjoint.

D_n and $\Delta - U_n$ are disjoint closed subsets. D_n has cardinality κ^+, so, by *, $|\Delta - U_n| \leq \kappa$. Hence $|\bigcap_{n\in\omega} U_n)| = \kappa^+$ and in particular Δ is not countably metacompact. This completes the proof of the Theorem. $\square$

References

[B1] Bešlagić A., *A Dowker product*, Trans. Amer. Math. Soc. **292** (1985), 519–530.

[B2] ______, *Another Dowker product*, Top. Appl. **36** (1990), 553–264.

[B3] ______, *Yet another Dowker product*, preprint.

[Dv] Devlin K., *Constructability*, Springer Verlag, Berlin, 1984.

[DJ] Dodd A.J., Jensen R., *The covering lemma for K*, Ann. Math. Logic **22** (1982), 1–30.

[Dk] Dowker C.H., *On countably paracompact spaces*, Canad. J. Math. **3** (1951), 219–224.

[E] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[F] Fleissner W.G., *The normal Moore space conjecture and large cardinals*, in: [KV], 733–760.

[G] Good C., *Large cardinals and small Dowker spaces*, to appear in Proc. Amer. Math. Soc.

[K] Kunen K., *Set Theory, An Introduction to Independence Proofs*, North Holland, Amsterdam, 1984.

[KV] Kunen K., Vaughan J.E. eds., *Handbook of Set-Theoretic Topology*, North Holland, Amsterdam, 1984.

[R] Rudin M.E., *Dowker spaces*, in: [KV], 761–780.

[RS] Rudin M.E., Starbird M., *Products with a metric factor*, Gen. Top. Appl. **5**, 235–248.

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