A note on Boolean algebras

Isaac Gorelic

Abstract. We show that splitting of elements of an independent family of infinite regular size will produce a full size independent set.

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Let us note that a family $\{b_\alpha : \alpha < \kappa\}$ of elements of a Boolean algebra is called independent if for any finite disjoint sets $I, J \subseteq K$ the meet

$$\bigwedge_{\alpha\in I} b_\alpha \ \wedge \ \bigwedge_{\beta\in J} (-b_\beta) \neq 0\,.$$

The following theorem gives a positive answer to a question raised by P. Koszmider.

Theorem. *Let κ be an infinite regular cardinal and suppose that in a Boolean algebra $\mathcal{A}$ there is an independent family $\{a_\alpha : \alpha < \kappa\}$ of size κ. Suppose also that we have $\{b_\alpha : \alpha < \kappa\}$, $\{c_\alpha : \alpha < \kappa\}$, subsets of $\mathcal{A}$, s.t. $\forall\,\alpha < \kappa\ b_\alpha \vee c_\alpha = a_\alpha, b_\alpha \wedge c_\alpha = 0$. Then there exist $I \in [\kappa]^\kappa$ and $\varphi : I \to B \cup C$ with $\varphi(\alpha) \in \{b_\alpha, c_\alpha\}$ such that $\{\varphi(\alpha) : \alpha \in I\}$ is independent in $\mathcal{A}$.*

Proof: We may assume that $\mathcal{A}$ is a field of sets, $\mathcal{A} \subset \mathcal{P}(X)$ for some set X. For every $x \in X$, define $f_x : \kappa \to 2$ by $f_x(\alpha) = 1 \Leftrightarrow x \in a_\alpha$. Let $F = \{f_x : x \in X\}$. Then F is a dense subspace of the Cantor Cube 2^κ.

Let $A_\alpha = \{f \in F : f(\alpha) = 1\} = \{f_x : x \in a_\alpha\}$, similarly $B_\alpha = \{f_x : x \in b_\alpha\}$, $C_\alpha = \{f_x : x \in c_\alpha\}$. Then $\{A_\alpha : \alpha < \kappa\}$ is an independent family of subsets of F (and of 2^κ) and $\forall\,\alpha\ A_\alpha = B_\alpha \dot\cup C_\alpha$.

We notice that it is sufficient to find an $I \in [\kappa]^\kappa$ and $\varphi \in \prod_{i\in I}\{B_i, C_i\}$ such that $\{\varphi(\alpha) : \alpha \in I\}$ is an independent family of subsets of F. These $I = \{i_\alpha : \alpha < \kappa\}$ and φ we will construct by induction on α so that if we stop at some stage $\alpha < \kappa$, we will have the required I and φ at once.

At a stage $\alpha < \kappa$ we have selected $I_\alpha = \{i_\beta : \beta < \alpha\}$ and $\varphi_\alpha \in \prod_{i\in I_\alpha}\{B_i, C_i\}$ so that, denoting by $\mathcal{K}_\alpha$ the set of all Boolean independence combinations from $\{\varphi(i) : i \in I_\alpha\}$, $\forall\, K \in \mathcal{K}_\alpha\ \bar{K} \supset$ some $U_K \leftarrow$ a clopen (basic) subset of 2^κ, and we fix the family $\mathcal{U}_\alpha = \{U_K : K \in \mathcal{K}_\alpha\}$. This is our induction hypothesis.

At the stage α we choose i_α and $\varphi(i_\alpha)$ to satisfy our inductive hypothesis on the larger sets

$$I_{\alpha+1}, \quad \mathcal{K}_{\alpha+1}, \quad \mathcal{U}_{\alpha+1}\,.$$

Suppose we cannot. Let $J \in [\kappa]^\kappa$ be disjoint from all indices of subbasic sets mentioned in the definitions of members of $\mathcal{U}_\alpha$. (So that e.g. $\forall U \in \mathcal{U}_\alpha\ U \restriction J = 2^J$). Then every i in J is a "bad" index, and for such i we must have

$$\exists\, K_1 \in \mathcal{K}_\alpha\ \exists K_2 \in \mathcal{K}_\alpha \text{ s.t.}$$

either $K_1 \cap B_i$ is nowhere dense in 2^κ or $K_2 \cap C_i$ is nowhere dense in 2^κ.

Then either

$$U_{K_1} \subset Int(\bar{K}_1) \subset \overline{C_i \cup (F \setminus A_i)}$$

or

$$U_{K_2} \subset Int(\bar{K}_2) \subset \overline{B_i \cup (F \setminus A_i)}\,,$$

and similarly for every i in J.

But since $|\mathcal{K}_\alpha| = |\,[\alpha]^{<\omega}\,| < \kappa$ and κ is regular, there is $I \in [J]^\kappa$, a fixed $K \in \mathcal{K}_\alpha$ and a function $\varphi \in \prod_{i\in I}\{B_i, C_i\}$ s.t. for every $i \in I$

$$U_K \subset Int(\bar{K}) \subset \overline{\varphi(i) \cup (F \setminus A_i)}\,.$$

Then $\{\varphi(i) : i \in I\}$ is an independent family of size κ.

Indeed, let L be a Boolean independence combination from this family, and let $\tilde{L}$ be the same combination with A_i's replacing $\varphi(i)$'s.

Then $\emptyset \neq \tilde{L} \cap U_K$ is an elementary basic open set in $F \subset 2^\kappa$ such that $(\tilde{L} \cap U_K) \setminus L$ is nowhere dense in F.

Hence $L \neq \emptyset$, as required. □

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References

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[Ko] Koppelberg S., *General Theory of Boolean Algebras*, in Handbook of Boolean Algebras vol. 1, J.D. Monk, R. Bonnet, eds., North Holland, 1989.

University of Toronto and The Centre for Theoretical Study, Ovocný trh 3, 116 36 Praha 1, Czech Republic

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