Continuous selections, G_δ-subsets of Banach spaces and usco mappings

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Abstract. Every l.s.c. mapping from a paracompact space into the non-empty, closed, convex subsets of a (not necessarily convex) G_δ-subset of a Banach space admits a single-valued continuous selection provided every such mapping admits a convex-valued usco selection. This leads us to some new partial solutions of a problem raised by E. Michael.

Keywords: set-valued mapping, lower semi-continuous, upper semi-continuous, selection, countable-dimensional space

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1. Introduction

The famous Michael Theorem [2, Theorem 3.2$''$] tells that every lower semi-continuous (l.s.c.) mapping from a paracompact space into the non-empty, closed, convex subsets of a Banach space admits a single-valued continuous selection. In the present paper we show that, in a more general situation, this is actually equivalent to the existence of "nice" set-valued selections. The following theorem will be proved.

Theorem 1.1. *Let X be a paracompact space, E be a Banach space, and let $Y \subset E$ be a G_δ-subset. Then, the following two conditions are equivalent:*

(a) *every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$ admits a single-valued continuous selection;*
(b) *every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$ admits a convex-valued usco selection.*

Here, $\mathcal{F}_c(Y) = \{S \in \mathcal{F}(Y) : S \text{ is convex}\}$, where $\mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed in } Y\}$ and $2^Y = \{S \subset Y : S \neq \emptyset\}$. A set-valued mapping $\varphi : X \to 2^Y$ is l.s.c. if $\varphi^{-1}(U) = \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$. A set-valued mapping $\theta : X \to 2^Y$ is upper semi-continuous (u.s.c.) if $\theta^{\#}(U) = \{x \in X : \theta(x) \subset U\}$ is open in X for every open $U \in Y$. A set-valued mapping $\theta : X \to 2^Y$ is called usco provided that it is u.s.c. and compact-valued simultaneously. A map $f : X \to Y$ (resp. $\theta : X \to 2^Y$) is a selection for φ if $f(x) \in \varphi(x)$ (resp. $\theta(x) \subset \varphi(x)$) for every $x \in X$.

Turning to the possible applications of Theorem 1.1, let us especially mention that this result is closely related to the following E. Michael's problem in [6]:

Problem 396. *Let X be paracompact, E be a Banach space, $Y \subset E$ a convex G_δ-subset, and let $\varphi : X \to \mathcal{F}_c(Y)$ be l.s.c.; Does there exists a single-valued continuous selection f for φ?*

What is known to this question in all, is that it has an affirmative answer if X is finite-dimensional [3, Theorem 1.2 and Example 2.5] or if Y is such that $\overline{\operatorname{conv}(K)} \subset Y$ for every compact $K \subset Y$ [4, Theorem 1.1] and [2, Propositions 2.6 and 2.3 and Theorem 3.2$''$]. But, in general, it is still open even if X is a compact metric space (see [6]).

Using Theorem 1.1 we now obtain two further partial results to this problem. First, let us recall that a space X is of countable dimension provided that it is a countable union of finite-dimensional subsets; A space X is strongly countable-dimensional provided it is a countable union of closed finite-dimensional subsets.

Corollary 1.2. *Let X be a countable-dimensional metric space, E be a Banach space, and let $Y \subset E$ be a G_δ-subset. Then every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$ admits a single-valued continuous selection.*

PROOF: Suppose $\varphi : X \to \mathcal{F}_c(Y)$ is l.s.c.. Since X is countable-dimensional and Y is completely metrizable, by [1, Theorem 2.1], φ admits a finite-valued u.s.c. selection ψ. Setting then $\theta(x) = \operatorname{conv}(\psi(x))$, we get a convex-valued usco selection θ for φ. Finally, Theorem 1.1 completes the proof. □

Corollary 1.3. *Let X be a strongly countable-dimensional paracompact space, E be a Banach space, and let $Y \subset E$ be a G_δ-subset. Then every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$ admits a single-valued continuous selection.*

PROOF: Following the previous proof, it suffices to show that every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$ admits a finite-valued u.s.c. selection. That this is so, it follows from [9, Theorem 4.5]. □

A construction of continuous selections avoiding F_σ-sets is exhibited in the next Section 2. A proof of Theorem 1.1 is obtained in the last Section 3.

2. A construction of continuous selections avoiding F_σ-sets

Throughout this section, (E, d) will denote a Banach space with a metric d generated by the norm of E, and $Y = \bigcap\{V_n : 1, 2, \dots\}$ where each $V_n \subset E$ is open. For $\varepsilon > 0$ and $F \in 2^E$, we use $B_\varepsilon(F)$ to denote the ε-neighbourhood of F in (E, d). If $\varphi : X \to 2^E$, we shall, for convenience, denote by $\bar{\varphi} : X \to \mathcal{F}(E)$ the mapping defined by $\bar{\varphi}(x) = \overline{\varphi(x)}$ (i.e. the closure of $\varphi(x)$ in E).

Lemma 2.1. *Let X be a topological space such that, for every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ and every n, there is a continuous selection $f : X \to V_n$ for $\bar{\varphi}$. Then every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a single-valued continuous selection.*

PROOF: Suppose $\varphi : X \to \mathcal{F}_c(Y)$ is l.s.c.. By our hypotheses, there is a continuous selection $f_1 : X \to V_1$ for $\bar{\varphi}$. Define a continuous mapping $r_1 : X \to (0, 1]$ by letting

$$r_1(x) = \min\{d(f_1(x), E \setminus V_1), 1\}, \quad x \in X .$$

By induction, we shall construct a sequence $\{f_n\}$ of continuous selections $f_n : X \to V_n$ for $\bar{\varphi}$ and a sequence $\{r_n\}$ of continuous maps $r_n : X \to (0, 1]$ such that, for every n and $x \in X$,

(1) $r_n(x) \le \min\{d(f_k(x), E \setminus V_k) : k \le n\}$, and
(2) $d(f_{n+1}(x), f_n(x)) \le 3^{-n} \cdot r_n(x)$.

This will be sufficient because, by (2), $\{f_n\}$ is a Cauchy sequence. So, it must converge to some continuous $f : X \to E$. By (1) and [8, Lemma 6.1.1], $f(x) \notin \bigcup\{E \setminus V_n : n = 1, 2, \dots\}$. That is, $f : X \to Y$ and therefore f is a selection for φ.

So, it only remains to define these f_n and r_n. Since f_1 and r_1 were defined above, we may suppose that $f_1, \dots, f_n$ and $r_1, \dots, r_n$ have already been defined, and we must define f_{n+1} and r_{n+1}. Define a set-valued mapping $\varphi_n : X \to \mathcal{F}_c(Y)$ by letting

$$\varphi_n(x) = \overline{B_{3^{-n} \cdot r_n(x)}(f_n(x)) \cap \varphi(x)}^{Y}, \quad x \in X .$$

Note, by [7, Lemma 2.2] and [2, Proposition 2.3], φ_n is l.s.c.. Then, by our hypotheses, there is a continuous $f_{n+1} : X \to V_{n+1}$ such that $f_{n+1}(x) \in \overline{\varphi_n(x)} \subset \overline{\varphi(x)}$ for every $x \in X$. Since $f_{n+1}(x) \in \overline{B_{3^{-n} \cdot r_n(x)}(f_n(x))}$, (2) holds. Defining finally $r_{n+1} : X \to (0, 1]$ by $r_{n+1}(x) = \min\{d(f_{n+1}(x), E \setminus V_{n+1}), r_n(x)\}$, we finish the proof. □

Corollary 2.2. *Let Y be such that, for every n and every $K \subset Y$ compact, $\overline{\operatorname{conv}(K)} \cap V_n$ is convex. Then every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$, with X a paracompact space, admits a single-valued continuous selection.*

Proof: Suppose $\varphi : X \to \mathcal{F}_c(Y)$ is l.s.c. and n is a positive integer. Define first a set-valued mapping $\psi : X \to 2^E$ by $\psi(x) = V_n$ for every $x \in X$. Next, define $\theta : X \to 2^E$ by $\theta(x) = \bar{\varphi}(x) \cap \psi(x)$. Notice, that ψ has an open graph in $X \times E$ and $\theta(x)$ is convex and non-empty for all $x \in X$. Then, by [5, Theorem 9.1], θ admits a continuous selection f. That is, there exists a continuous $f : X \to V_n$ which is a selection for $\bar{\varphi}$. Applying finally Lemma 2.1, we complete the proof. □

3. Proof of Theorem 1.1.

In preparation for the proof of Theorem 1.1 we begin by proving the following

Lemma 3.1. *Let X be a paracompact space, (E, d) a metric space, $\varphi : X \to 2^E$ l.s.c., and let θ be an usco selection for φ. Then, for every open cover $\mathcal{W}$ of X and every map $\delta : X \to (0, +\infty)$ there exist*

(1) *a locally-finite open cover $\mathcal{U}$ of X,*
(2) *a map $u : \mathcal{U} \to \mathcal{W}$, and*
(3) *a map $\varkappa : \mathcal{U} \to X$,*

such that

(a) $\mathcal{U}' \subset \mathcal{U}$ *and* $\bigcap \mathcal{U}' \neq \emptyset$ *implies* $\varkappa(\mathcal{U}') \subset \bigcap u(\mathcal{U}') = \bigcap\{u(U) : U \in \mathcal{U}'\}$,
(b) $\theta(z) \subset B_{\delta(\varkappa(U))}(\theta(\varkappa(U)))$, $z \in U \in \mathcal{U}$,
(c) $\theta(\varkappa(U)) \subset B_{\delta(\varkappa(U))}(\varphi(z))$, $z \in U \in \mathcal{U}$.

PROOF: Since X is a paracompact space, by [3, Lemma 11.4], there is an open cover $\mathcal{V}$ of X and a map $v : \mathcal{V} \to \mathcal{W}$ such that

$$(*) \qquad \mathcal{V}' \subset \mathcal{V} \text{ and } \bigcap \mathcal{V}' \neq \emptyset \text{ implies } \bigcup \mathcal{V}' \subset \bigcap v(\mathcal{V}')\,.$$

Define, in a natural fashion, a map $S : X \to \mathcal{V}$ such that, for every $x \in X$, $x \in S(x)$. Next, for every $x \in X$, we set

$$G_x = \{z \in S(x) \cap \theta^{\#}(B_{\delta(x)}(\theta(x))) : \theta(x) \subset B_{\delta(x)}(\varphi(x))\}\,.$$

Since $\theta(x)$ is compact and since φ is l.s.c., by [3, Lemma 11.3], G_x is a neighbourhood of x. So, $\{G_x : x \in X\}$ is an open cover of X refining $\mathcal{V}$. Let, then, $\mathcal{U}$ be a locally-finite open cover of X which refines $\{G_x : x \in X\}$. For every $U \in \mathcal{U}$ pick a fixed point $\varkappa(U) \in X$ such that $U \subset G_{\varkappa(U)}$, and then define $u : \mathcal{U} \to \mathcal{W}$ by

$$u(U) = v(S(\varkappa(U))), \quad U \in \mathcal{U}\,.$$

These $\mathcal{U}$, u and $\varkappa$ satisfy all our requirements. In fact, we have only to check (a). Suppose $\mathcal{U}' \subset \mathcal{U}$ with $\bigcap \mathcal{U}' \neq \emptyset$. Then $\varkappa(U) \in S(\varkappa(U))$ and $U \subset G_{\varkappa(U)} \subset S(\varkappa(U))$, $U \in \mathcal{U}'$, implies $\varkappa(\mathcal{U}') \subset \bigcup\{S(\varkappa(U)) : U \in \mathcal{U}'\} \subset \bigcap\{v(S(\varkappa(U))) : U \in \mathcal{U}'\} = \bigcap u(\mathcal{U}')$ (see $(*)$), which completes the proof. $\square$

Having established Lemma 3.1, we now proceed to the proof of Theorem 1.1. In fact, we have only to prove (b) $\to$(a). Suppose $Y = \bigcap\{V_n : n = 1, 2, \dots\}$, where each $V_n \subset E$ is open. Pick a fixed n, and let $\varphi : X \to \mathcal{F}_c(Y)$ be l.s.c.. By virtue of Lemma 2.1, it suffices to construct a continuous selection $f : X \to V_n$ for $\bar{\varphi}$. Towards this end, let θ be a convex-valued usco selection for φ, which exists by virtue of (b).

Define $\varrho : X \to (0, +\infty)$ by $\varrho(x) = d(\theta(x), E \setminus V_n)$, $x \in X$. Note that this definition is correct because $\theta(x) \subset \varphi(x) \subset V_n$ and because $\theta(x)$ is compact. Next, let $\mathcal{W}$ be a locally-finite open cover of X and let $\eta : \mathcal{W} \to X$ be such that

$$(3.2) \qquad \theta(x) \subset B_{\varrho(\eta(W))}(\theta(\eta(W))) \subset V_n \ \text{ for every } \ x \in W \in \mathcal{W}.$$

Such $\mathcal{W}$ and η can be obtained by using, for instance, Lemma 3.1.

Next, define another map $\delta : X \to (0, +\infty)$ by

$$\delta(x) = \min\left\{\frac{1}{2} d(\theta(x), E \setminus B_{\varrho(\eta(W))}(\theta(\eta(W)))) : W \in \mathcal{W},\ x \in W\right\}, \quad x \in X.$$

This is possible because θ is compact-valued and $\mathcal{W}$ is locally-finite. Let us note the following property of δ:

$$B_{2\delta(x)}(\theta(x)) \subset B_{\varrho(\eta(W))}(\varrho(\eta(W))) \text{ for every } x \in W \in \mathcal{W}. \tag{3.3}$$

Let now $\mathcal{U}$, $u : \mathcal{U} \to \mathcal{W}$ and $\varkappa : \mathcal{U} \to X$ be as in Lemma 3.1 (applied with these particular φ, θ, $\mathcal{W}$ and δ). Because of the paracompactness of X, there is an open cover $\{G_U : U \in \mathcal{U}\}$ of X such that $\overline{G_U} \subset U$ for every $U \in \mathcal{U}$. Whenever $U \in \mathcal{U}$, we define a set-valued mapping $\varphi_U : \overline{G_U} \to 2^E$ by letting

$$\varphi_U(z) = \overline{\varphi(z) \cap B_{\delta(\varkappa(U))}(\theta(\varkappa(U)))}, \quad z \in \overline{G_U}\,.$$

The following holds:

$$\varphi_U(z) \in \mathcal{F}_c(E)\,. \tag{3.4}$$

Indeed, by (c) of Lemma 3.1, $\varphi(z) \cap B_{\delta(\varkappa(U))}(\theta(\varkappa(U))) \neq \emptyset$. Then, (3.4) follows from the convexity of $\varphi(z)$ and $\theta(\varkappa(U))$.

(3.5) $\varphi_U(z) \subset \bar{\varphi}(z)$, which follows immediately from the definition of φ_U.

(3.6) φ_U is l.s.c. because $\varphi|\overline{G_U}$ is l.s.c. and because $B_{\delta(\varkappa(U))}(\theta(\varkappa(U)))$ is open (see, [2, Propositions 2.3 and 2.4]).

Now, by (3.4) and (3.6), making use of [2, Theorem 3.2$''$], we get a continuous selection $f_U : \overline{G_U} \to E$ for φ_U. Let $\{g_U : U \in \mathcal{U}\}$ be a partition of unity on X indexed-subordinated to $\{G_U : U \in \mathcal{U}\}$. We finally define $f : X \to E$ by letting

$$f(x) = \sum\{g_U(x) \cdot f_U(x) : U \in \mathcal{U}\}\,,$$

and let us check that f is the required one. Since φ is convex-valued, by (3.5), f is a selection for $\bar{\varphi}$. So, it only remains to check that $f(X) \subset V_n$. Towards this end, let $x \in X$. Set $\mathcal{U}_x = \{U \in \mathcal{U} : x \in G_U\}$. Note that $g_U(x) \neq 0$ implies $U \in \mathcal{U}_x$. Pick a fixed $W \in u(\mathcal{U}_x)$. Then, by (a) of Lemma 3.1, we get that

$$\varkappa(\mathcal{U}_x) \subset \bigcap u(\mathcal{U}_x) \subset W\,.$$

Together with (3.3), this leads us to the inclusions

$$\begin{aligned} f_U(x) \in \overline{B_{\delta(\varkappa(U))}(\theta(\varkappa(U)))} \subset B_{2\cdot\delta(\varkappa(U))}(\theta(\varkappa(U))) \subset \\ \subset B_{\varrho(\eta(W))}(\theta(\eta(W))), \quad U \in \mathcal{U}_x\,. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \sum\{g_U(x) \cdot f_U(x) : U \in \mathcal{U}\} = \\ &= \sum\{g_U(x) \cdot f_U(x) : U \in \mathcal{U}_x\} \subset B_{\varrho(\eta(W))}(\theta(\eta(W))) \subset V_n\,. \end{aligned}$$

Thus, the proof of Theorem 1.1 is completed. □

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