On the Jacobson radical of strongly group graded rings

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Abstract. For any non-torsion group G with identity e, we construct a strongly G-graded ring R such that the Jacobson radical $J(R_e)$ is locally nilpotent, but $J(R)$ is not locally nilpotent. This answers a question posed by Puczyłowski.

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Several interesting results of ring theory establish the local nilpotency of the Jacobson radical of some ring constructions (cf. [9]). In this paper we consider an analogous question for strongly group graded rings. Let G be a group. An associative ring $R = \bigoplus_{g\in G} R_g$ is said to be *strongly G-graded* if $R_gR_h = R_{gh}$ for all $g, h \in G$. Strongly group graded rings have been intensively investigated for several years (cf., for example, [12],[15],[20]). In [18] the following question was posed: is it true that for every free group G of rank ≥ 2 the Jacobson radical of each strongly G-graded ring is locally nilpotent? (As it is noted in [18], the question is also connected with [14], Problem 24, and with a problem on the local nilpotency of the Jacobson radical of a skew polynomial ring, cf. [19].) It follows from the results of [6] that the answer is positive in the case when R_e satisfies the ascending chain condition for left annihilators, where e is the identity of G. It is also known that the answer is positive for group rings of free groups of rank ≥ 2 (cf. [18]). The answer to the analogous question for the rings of polynomials in at least two non-commuting variables is also positive (cf. [18]).

We shall show that in general the answer is negative. Namely, for an arbitrary group G, we construct a strongly G-graded ring R such that the Jacobson radical $J(R)$ is not nil. On the other hand, we shall prove that, for the positive answer to the question above, it suffices to assume that $J(R_e)$ is left T-nilpotent. It will also be shown that the weaker condition that $J(R_e)$ is equal to the Baer radical $B(R_e)$, is not sufficient for the local nilpotency of $J(R)$.

Our proofs are based on the previous results of [7], [10] and [21].

Theorem 1. *For each group G, there exists a strongly G-graded ring R such that the Jacobson radical $J(R)$ is not nil.*

Lemma 1. *Let* $R = \bigoplus_{g\in G} R_g$ *be a* G*-graded ring, and let* $h \in G$*. Then there exists a* G*-graded ring* $Q = \bigoplus_{g\in G} Q_g$ *such that* $Q \supseteq R, J(Q) \supseteq J(R), Q_g \supseteq R_g$ *and* $Q_gQ_h \supseteq R_{gh}$ *for each* $g \in G$*.*

Proof: Let Z be the ring of integers, R^1 the ring R with identity 1 adjoined, $Z[x,y]$ the ring of polynomials with non-commuting variables x, y. Denote by W the free product of R and $Z[x,y]$. For $w \in W$, let $\langle w\rangle$ be the subring generated in W by w. Put $M = R + Ry + xR + xRy$, $S = M + \langle xy\rangle + \langle xy\rangle x + y\langle xy\rangle + \langle yx\rangle$. To simplify the notation, we shall denote by the same letters elements and their images in the quotient rings which will be introduced. If we factor out the ideal generated in W by x^2, y^2, yR, Rx and all $r - yxr$, $r - ryx$, where r runs over R, then the resulting quotient ring Q is equal to $Z + \langle x\rangle + \langle y\rangle + S$. Clearly, S and M are ideals of Q. It is routine to verify that

$$M = \begin{bmatrix} R & Ry \\ xR & xRy \end{bmatrix}$$

and

$$S/M = \begin{bmatrix} \langle xy\rangle & \langle xy\rangle x \\ y\langle xy\rangle & \langle yx\rangle \end{bmatrix}$$

are Morita contexts (cf. [1]). Further, $R, xRy \cong R$ and $Ry, xR \cong R^0$, where R^0 stands for the ring with zero multiplication defined on the additive group of R. Since $\langle xy\rangle$ and $\langle yx\rangle$ are semiprime rings and S/M satisfies the left annihilator condition in the sense of [21], then [21], Lemma 2.6, implies that S/M is semiprime. Therefore $J(S) = J(M)$.

Take any $q \in J(Q)$, say $q = a + bx + cy + s$, where $a, b, c \in Z$, $s \in S$. If $a \neq 0$, then $qxy \notin M$ and so $0 \neq qxy \in J(S/M)$, a contradiction. If $a = 0, b \neq 0$, then $0 \neq qy \in J(S/M)$ gives a contradiction. Finally, if $a = b = 0$, $c \neq 0$, then $0 \neq qx \in J(S/M)$, a contradiction again. Therefore $a = b = c = 0$, that is $q \in J(M)$. Thus $J(Q) = J(M)$.

Denote by I the ideal generated in Q by $J(R)$. Then

$$I = \begin{bmatrix} J(R) & J(R)y \\ xJ(R) & xJ(R)y \end{bmatrix}.$$

Clearly, I is the largest ideal of M satisfying the property that $I \cap R \subseteq J(R)$ and $I \cap xRy \subseteq J(xRy) = xJ(R)y$. In view of [10], Corollary 1, and [11], Corollary 6, we conclude $I = J(M)$. Hence $I = J(Q)$. In particular, $J(Q) \supseteq J(R)$.

To make Q a G-graded ring, we put $x \in Q_h$, $y \in Q_{h^{-1}}$, $Z \subseteq Q_e$, and then the grading naturally comes from R. For example, $xR_gy \subseteq Q_{hgh^{-1}h} \subseteq Q_{hg}$. Since $R_{gh}y \subseteq Q_g$ and $x \in Q_h$, we get $Q_gQ_h \supseteq (R_{gh}y)x = R_{gh}$, as required. □

Lemma 2. *Let* $R = \bigoplus_{g \in G} R_g$ *be a* G*-graded ring. Then there exists a* G*-graded ring* $Q = \bigoplus_{g \in G} Q_g$ *such that* $Q \supseteq R$, $J(Q) \supseteq J(R)$, $Q_g \supseteq R_g$ *and* $Q_g Q_h \supseteq R_{gh}$ *for all* $g, h \in G$.

Proof: Denote by $R^{(h)}$ the ring constructed by R and h in Lemma 1. We may order the set G, identify the elements of G with ordinal numbers and define an ascending chain of G-graded rings T_α by putting $T_1 = R^{(1)}, T_\alpha = (\bigcup_{\beta<\alpha} T_\beta)^{(\alpha)}$. The transfinite induction shows that $J(T_\alpha) \supseteq J(R)$ in view of Lemma 1. However, $G = \{\alpha | \alpha \leq \tau\}$ for some τ. Hence a straightforward verification shows that $Q = \bigcup_{\alpha \leq \tau} T_\alpha$ is the desired ring. □

Lemma 3. *Let* $R = \bigoplus_{g \in G} R_g$ *be a* G*-graded ring. Then there exists a strongly* G*-graded ring* $Q = \bigoplus_{g \in G} Q_g$ *such that* $Q \supseteq R$, $J(Q) \supseteq J(R)$, *and* $Q_g \supseteq R_g$ *for all* $g \in G$.

Proof: Denote by R' the ring constructed in Lemma 2, and put $R^{[1]} = R'$, $R^{[n+1]} = (R^{[n]})'$. Then it is routine to verify that $Q = \bigcup_{n=1}^{\infty} R^{[n]}$ is the required example. □

Proof of Theorem 1 easily follows from Lemma 3 if we take any quasi-regular but not nil ring R and make it G-graded with $R_e = R$.

Now we shall give a new condition sufficient for the Jacobson radical of a ring strongly graded by a free group to be locally nilpotent. In fact, our condition is applicable not only to free groups, but also to all u.p.-groups. A group G is called a *unique product* (u.p.-)*group* if, for any non-empty subsets X, Y of S, there exists at least one element uniquely presented in the form xy, where $x \in X$, $y \in Y$ (cf. [16], § 13.1). The radicals of rings graded by u.p.-groups were considered, in particular, in [6] and [7]. A ring R is said to be *left* T*-nilpotent* if, for every sequence $x_1, x_2, \ldots \in R$, there exists n such that $x_1 \ldots x_n = 0$. The class of left T-nilpotent rings lies strictly between the class of nilpotent rings and the Baer radical class (cf. [5]).

Theorem 2. *Let* G *be a u.p.-group,* $R = \bigoplus_{g \in G} R_g$ *a strongly* G*-graded ring. If* $J(R_e)$ *is left* T*-nilpotent, then* $J(R)$ *is locally nilpotent.*

Proof: Given that G is a u.p.-group, it follows from [7], Theorem 2.2, that the Levitzky radical $L(R)$ is homogeneous, i.e. $L(R) = \bigoplus_{g \in G} L(R) \cap R_g$. Since $R/L(R)$

is strongly G-graded, we may assume that from the very beginning $L(R) = 0$.

Suppose to the contrary that $J(R) \neq 0$. For $r \in R$, $g \in G$, denote by r_g the projection of r on R_g, and put $\operatorname{supp}(r) = \{g \in G | r_g \neq 0\}$. Let $l(r) = |\operatorname{supp}(r)|$. Choose a non-zero element s in $J(R)$ with minimal length $l(s)$, and take any $h \in \operatorname{supp}(s)$. If $s_h R_{h^{-1}} = 0$, then $s_h R_{h^{-1}} R = 0$, and so $s_h \in A = \{r \in R | rR = 0\}$. However, $A \subseteq L(R) = 0$, because $A^2 = 0$. Thus $s_h R_{h^{-1}} \neq 0$. Therefore the set $P = \{r_e | r \in J(R),\ l(r) = l(s)\}$ is non-zero. Given that G is a u.p.-group, Theorem 3.2 of [7] tells us that $P \subseteq J(R_e)$. Denote by I the ideal generated in R by P. We claim that I is left T-nilpotent.

Suppose that there exists a sequence of elements $x_1, x_2, \dots$ of I such that $x_1 \dots x_n \neq 0$ for all n. Each x_i is a finite sum of elements of the form $a r_e b$, where $r \in J(R),\ l(r) = l(s)$, a and b are homogeneous elements of R^1. We may assume that all b belong to R. (Indeed, if $b \in Z$, then we can replace x_i by $x_i x_{i+1}$, and consider the sequence $x_1, \dots, x_i x_{i+1}, x_{i+2}, \dots$.) Denote by $S(x_i)$ the set of such summands of x_i. For arbitrarily large n we can pick $y_1 \in S(x_1), \dots, y_n \in S(x_n)$ such that $y_1 \dots y_n \neq 0$. Since all the $S(x_i)$ are finite, a standard argument shows that there exists an infinite sequence $y_1, y_2, \dots$ where $y_i \in S(x_i)$ and $y_1 \dots y_n \neq 0$ for all n. Then $y_i = a^{(i)} r_e^{(i)} b^{(i)}$ where $r^{(i)} \in J(R),\ l(r^{(i)}) = l(s),\ a^{(i)}$ and $b^{(i)}$ are homogeneous elements of R^1, and $b^{(i)} \in R$. Given that R is strongly graded, $b^{(2)} \in R$, and G is a group, it follows that $b^{(2)} = c^{(2)} d^{(2)}$ for some homogeneous elements $c^{(2)}, d^{(2)}$ such that $b^{(1)} a^{(2)} r_e^{(2)} c^{(2)} \in R_e$. Similarly, for any $i \geq 3$, there exist homogenous elements $c^{(i)}, d^{(i)}$ such that $b^{(i)} = c^{(i)} d^{(i)}$ and $d^{(i-1)} a^{(i)} r_e^{(i)} c^{(i)} \in R_e$. Let $z_1 = r_e^{(1)}, z_2 = b^{(1)} a^{(2)} r_e^{(2)} c^{(2)}$, and $z_i = d^{(i-1)} a^{(i)} r_e^{(i)} c^{(i)}$ for $i \geq 3$. Then, $z_1, z_2, z_3, \dots \in P$. Since $J(R_e)$ is left T-nilpotent and contains P, we get $z_1 \dots z_n = 0$ for some $n > 1$. Hence $y_1 \dots y_n = a^{(1)} z_1 \dots z_n d^{(n)} = 0$.

This contradiction shows that I is left T-nilpotent, and so $I \subseteq L(R) = 0$. Therefore $J(R) = 0$, which completes the proof. $\square$

Let $\mathbb{P}$ denote the set of positive integers. The well-known Golod's example of a nil but not locally nilpotent ring R is $\mathbb{P}$-graded (cf. [16]). Therefore one cannot replace strongly graded rings by arbitrary graded rings in Theorem 2. Now we shall show that the left T-nilpotence cannot be weakened to Baer radicalness, either.

Theorem 3. *Let G be a non-periodic group with identity e. Then there exists a strongly G-graded ring Q such that $J(Q_e) = B(Q_e)$ but $J(Q) \neq L(Q)$.*

Proof: Let R be the Golod ring. Since R is $\mathbb{P}$-graded, R can easily be made G-graded with $R_e = 0$. Take any $h \in G$ and denote by Q, S, M the rings constructed by R as in the proof of Lemma 1. It has been proved that $J(Q) = J(M)$. The same reasoning shows that $B(Q) = B(M)$. Further,

$$M \cong \begin{bmatrix} R & Ry \\ xR & xRy \end{bmatrix},$$

whence

$$M_e = \begin{bmatrix} R_e & R_h y \\ x R_{h^{-1}} & x R_e y \end{bmatrix}.$$

Evidently R_e and xR_ey are isomorphic to R_e which satisfies $J(R_e) = B(R_e)$, because it is zero. It follows from [10], Corollary 1, and [11], Corollary 6, that $J(M_e)$ is equal to the largest ideal I of M_e with the property that $I \cap R_e \subseteq J(R_e)$ and $I \cap xR_ey \subseteq J(xR_ey)$. Besides, [10], Corollary 3, and [11], Corollary 6, imply that $B(M_e)$ is the largest ideal of M_e with the property that $I \cap R_e \subseteq B(R_e)$ and $I \cap xR_ey \subseteq B(xR_ey)$. Therefore $J(M_e) = B(M_e)$. Further, $Q_e/S_e \cong Z$ and $S_e/M_e = \langle xy \rangle + \langle yx \rangle$ imply $J(Q_e) = B(M_e)$. Thus $J(Q_e) = B(Q_e)$. It follows from [21], Lemma 2.3, that $J(Q_e) \supseteq J(R_e)$. This and transfinite induction show that all rings Q obtained from R in Lemmas 2 and 3 satisfy $J(Q_e) = B(Q_e)$. However $J(Q)$ is not locally nilpotent, because $J(Q) \supseteq J(R)$. Thus Q is the required example. □

Note that in the opposite case, where G is locally finite, it follows from the results of [2] and [3] that $J(R_e) = L(R_e)$ implies $J(R) = L(R)$ (cf. [13], Lemma 1.1). Analogous results were obtained in [2] for the more general case of rings graded by locally finite semigroups. A sufficient condition for the Jacobson radical of an algebra graded by a finite group to be nilpotent follows from the main theorem of [17].

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