M-mappings make their images less cellular

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Abstract. We consider M-mappings which include continuous mappings of spaces onto topological groups and continuous mappings of topological groups elsewhere. It is proved that if a space X is an image of a product of Lindelöf Σ-spaces under an M-mapping then every regular uncountable cardinal is a weak precaliber for X, and hence X has the Souslin property. An image X of a Lindelöf space under an M-mapping satisfies $cel_\omega X \leq 2^\omega$. Every M-mapping takes a $\Sigma(\aleph_0)$-space to an $\aleph_0$-cellular space. In each of these results, the cellularity of the domain of an M-mapping can be arbitrarily large.

Keywords: M-mapping, topological group, Maltsev space, $\aleph_0$-cellularity

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1. Introduction

We define the notion of an M-mapping that takes its origin from topological groups and Maltsev spaces (see [7], [18] for the discussion of Maltsev spaces). Then we prove the results mentioned above thus generalizing [11, Theorem 1], [16, Theorem 1] and [9, Theorem 1.1].

Let G be a topological group. The mapping $F : G^3 \to G$ defined by $F(x, y, z) = x \cdot y^{-1} \cdot z$ is continuous and satisfies the condition

$$(*) \qquad F(x, y, y) = F(y, y, x) = x \text{ for all } x, y \in G.$$

This reason gives rise for the notion of Maltsev spaces [7], [18], or M-spaces for short: we say that X is an <u>M-space</u>, if there exists a continuous mapping $F : X^3 \to X$ satisfying $(*)$. The mapping F is called a <u>Maltsev operation</u> on X. It is clear that every topological group is an M-space. In fact, every retract of a topological group is an M-space (if r is a continuous retraction of a topological group G onto its subspace X, define $F : X^3 \to X$ by $F(x, y, z) = r(x \cdot y^{-1} \cdot z)$ for all x, y, z of X). It is an open problem whether every M-space is a retract of some topological group.

Every σ-compact topological group has the Souslin property [11]. Uspenskiĭ [17] generalized this result by proving that any Maltsev Lindelöf Σ-space has the same property (and even is $\aleph_0$- cellular). In fact, a little bit more is proved in [11]: if γ is a family of open sets in a σ-compact topological group G and the cardinality of γ is uncountable and regular, then there exists a subfamily μ of γ such that $|\mu| = |\gamma|$ and $U \cap V \neq \emptyset$ for all U, V of μ. In other words, every uncountable regular cardinal is a weak precaliber for G. The same conclusion can be proved for a Lindelöf Σ-group or more generally, for a Maltsev Lindelöf Σ-space.

Our aim is to show that both the Souslin property and the weak precaliber property arise when the structures of an M-space and of a Lindelöf Σ-space are 'separated' by a continuous mapping of a special kind. In fact, the structure of an M-space will be completely replaced by the so-called M-mapping.

1.1 Definition. *We call $f : X \to Y$ an M-mapping provided that there exists a continuous mapping $F : X^3 \to Y$ such that $F(x, y, y) = F(y, y, x) = f(x)$ for all x, y of X.*

It is clear that every M-mapping is continuous. If $f : X \to Y$ is continuous and X (or Y) is an M-space, then f is an M-mapping. One can easily see that if every continuous mapping of a space X to elsewhere is an M-mapping, then X is an M-space. Furthermore, X is an M-space iff the identity mapping id_X is an M-mapping. Also note that $f : X \to Y$ is an M-mapping if there exist an 'intermediate' M-space Z and continuous mappings $g : Z \to Y$ and $h : X \to Z$ such that $f = g \circ h$. The class of M-mappings does not consist only of mappings generated by M-spaces: it is not necessary to require that g be defined on the whole space Z; it suffices to assume that g is defined on a subspace Z_0 of Z containing the set $F_0(h(X)^3)$, where F_0 is a Maltsev operation for Z. Indeed, define a continuous mapping $F : X^3 \to Y$ by $F(x, y, z) = gF_0(h(x), h(y), h(z))$ for all x, y, z of X. Obviously, F is as in Definition 1.1. Note that neither $h(X)$ nor Z_0 have to be M-subspaces of Z. This is the reason why the notion of an M-mapping seems to be flexible. Some additional information on M-mappings is contained in [13].

We denote by $w(X)$, $l(X)$ and $c(X)$ the weight, Lindelöf number and cellularity of a space X respectively. A G_τ-set in X is an intersection of at most τ many open subsets of X. The τ-cellularity of X, denoted by $cel_\tau X$, is the least cardinal λ such that every family γ of G_τ-sets in X contains a subfamily μ of cardinality less than or equal to λ satisfying $cl(\cup\mu) = cl(\cup\lambda)$. A space X is said to be τ-cellular if $cel_\tau X \leq \tau$.

All considered spaces are assumed to be completely regular.

The notion of a Σ-space (see [8]) is one of the most important in the paper. For the reader's convenience we give some definitions related to this concept.

1.2 Definition. *We call X a Σ-space, if there are two covers $\mathcal{K}$ and $\mathcal{C}$ of X such that $\mathcal{K}$ is σ-locally finite and consists of closed sets, $\mathcal{C}$ consists of countably compact sets and for every $C \in \mathcal{C}$ and every neighborhood U of C there exists $K \in \mathcal{K}$ satisfying $C \subseteq K \subseteq U$. If in addition $|\mathcal{K}| \leq \tau$, we call X a $\Sigma(\tau)$-space.*

Note that every countably compact space is obviously a $\Sigma(\aleph_0)$-space. The classes of Σ- and $\Sigma(\tau)$-spaces are closed with respect to countable unions and passing to perfect (even quasiperfect) images [8].

The following cardinal invariant was defined by Arhangel'skiĭ [1] in a slightly different form. Let βX be the Čech-Stone compactification of X and $\mathcal{F}$ the family of all closed subsets of βX.

1.3 Definition. *The* *Nagami number* *of a space X is the cardinal number*

$$Nag(X) = \min\{|\gamma| : (\gamma \subseteq \mathcal{F})\&(\forall x \in X\ \forall y \in \beta X \setminus X\ \exists F \in \gamma \text{ with } x \in F \not\ni y)\}.$$

In other words, the Nagami number of X is the least cardinality of a family $\gamma \subseteq \mathcal{F}$ separating the points of X from the points of $\beta X \setminus X$. Note that a Σ-space X is Lindelöf iff $Nag(X) \leq \aleph_0$. Lindelöf Σ-spaces can also be characterized as continuous images of spaces that admit perfect mappings onto separable metric ones [1]. Compact and σ-compact spaces constitute proper subclasses of the class of Lindelöf Σ-spaces, and the latter is countably productive. Clearly, $l(X) \leq Nag(X)$ for every space X. It needs mentioning that the Čech-Stone compactification βX in Definition 1.3 can be replaced by any compactification of X.

2. Main results

Let $\Pi = \prod_{i \in J} P_i$ be a product space and $p^* \in \Pi$. Denote by $\sigma(p^*)$ the subspace of Π consisting of all points which differ from p^* on at most finitely many coordinates. The following theorem presents the main result of the paper.

2.1 Theorem. *Let $\Pi = \prod_{i \in J} P_i$ be a product of spaces satisfying $Nag(P_i) \leq \tau$ for each $i \in J$, and Z be a subspace of Π such that $\sigma(p^*) \subseteq Z$. If X is an image of Z under an M-mapping, then every regular cardinal $\lambda > \tau$ is a weak precaliber for X.*

We will prove the theorem by making use of two auxiliary set-theoretic lemmas. The first of them reminds of the Δ-lemma [5, Appendix 2] and has a similar proof.

2.2 Lemma. *Suppose that λ and τ are infinite cardinals such that $cf(\lambda) > \tau^+$ and $k^\tau < \lambda$ for each $k < \lambda$. If ξ is a family of sets of cardinality less than or equal to τ and $|\xi| = \lambda$, then ξ contains a quasidisjoint subfamily η of the same cardinality λ, i.e. there exists a set R such that $M \cap N = R$ whenever $M, N \in \eta$ and $M \neq N$.*

Proof: For an arbitrary set A put $\xi_A = \{M \setminus A : M \in \xi\}$. It suffices to consider the case when the family ξ_A does not contain disjoint subfamilies of cardinality λ whenever $|A| < \lambda$. If $\alpha < \tau^+$ and a subfamily $\eta(\beta)$ of ξ with $|\eta(\beta)| < \lambda$ has already been defined for each $\beta < \alpha$, we put $B(\beta) = \cup\eta(\beta)$, $A(\alpha) = \bigcup_{\beta<\alpha} B(\beta)$ and define $\eta(\alpha)$ as a maximal disjoint subfamily of $\xi_{A(\alpha)}$. Obviously $|A(\alpha)| < \lambda$, and hence $|\eta(\alpha)| < \lambda$ as well. If now $M \in \xi \setminus \bigcup_{\alpha<\tau^+} \eta(\alpha)$, then $M \cap B(\alpha) \neq \emptyset$ for each $\alpha < \tau^+$. The latter implies $|M| > \tau$, for $B(\alpha) \cap B(\beta) = \emptyset$ whenever $\alpha \neq \beta$, which contradicts the assumption on the family ξ. $\square$

If γ is a cover of a set X and $x \in X$, we denote $St(x, \gamma) = \cup\{U \in \gamma : x \in U\}$.

2.3 Lemma (see [11]). *Let $\{\gamma_\alpha : \alpha < \lambda\}$ be a family of finite covers of a set T, where λ is a regular uncountable cardinal. Then for every sequence $\{x_\alpha : \alpha < \lambda\} \subseteq T$ there exists a subset $A \subseteq \lambda$ such that $|A| = \lambda$ and $St(x_\alpha, \gamma_\beta) \cap St(x_\beta, \gamma_\alpha) \neq \emptyset$ for all $\alpha, \beta \in A$.*

Proof of Theorem 2.1. Let f be an M-mapping of Z onto X and suppose that a continuous mapping $F : Z^3 \to X$ witnesses that. Suppose also that $\{O_\alpha : \alpha < \lambda\}$ is a family of non-void open sets in X, where λ is a regular cardinal, $\lambda > \tau$. Since $\sigma = \sigma(p^*)$ is dense in Π (and in Z), for each $\alpha < \lambda$ one can find a point $x_\alpha \in \sigma$ with $f(x_\alpha) \in O_\alpha$. For all $x \in \sigma$ and $\alpha < \lambda$ define $U_\alpha(x) \ni x$ as a maximal open set in σ such that

(1) $F(x_\alpha, y, z) \in O_\alpha$ and $F(z, y, x_\alpha) \in O_\alpha$ for all $y, z \in U_\alpha(x)$.

Then define $\gamma_\alpha = \{U_\alpha(x) : x \in \sigma\}$; $\alpha < \lambda$. It is clear that γ_α is an open cover of σ for each $\alpha < \lambda$. Our aim is to show that the conclusion of Lemma 2.3 holds for the family $\{\gamma_\alpha : \alpha < \lambda\}$ and the sequence $\{x_\alpha : \alpha < \lambda\}$. The further reasoning is divided into three parts.

Case 1. Suppose that $\lambda \leq 2^\tau$. For each $x \in \sigma$ denote by $supp(x)$ the set of all indices of J on which x differs from p^*. Put $B = \bigcup_{\alpha<\lambda} supp(x_\alpha)$. Then $|B| \leq \lambda$, because $|supp(x)| < \aleph_0$ whenever $x \in \sigma$. Denote by π_B the projection of Π onto $\Pi_B = \prod_{i\in B} P_i$. The subspace $\sigma_B = \{x \in \sigma : supp(x) \subseteq B\}$ of Π is homeomorphic to $\pi_B(\sigma)$, and the latter is the σ-product $\sigma(\pi_B(p^*)) \subseteq \Pi_B$ with the base point $\pi_B(p^*)$. We have $|B| \leq \lambda \leq 2^\tau$ and $Nag(P_i) \leq \tau$ for all $i \in B$. Therefore $Nag(\sigma_B) = Nag(\pi_B(\sigma)) \leq \tau$ by a theorem of Ranchin [2]. Let $\beta\Pi$ be the Čech-Stone compactification of Π and $\mathcal{F}$ a family of closed sets in $\beta\Pi$ separating points of σ_B from the points of $\beta\Pi \setminus \sigma_B$, $|\mathcal{F}| \leq \tau$. We can assume that the family $\mathcal{F}$ is closed under finite intersections. For every $\alpha < \lambda$ denote by $\tilde{\gamma}_\alpha$ a family of open sets in $\beta\Pi$ whose traces on σ constitute the cover γ_α of σ and find an element $L \in \mathcal{F}$ such that $x_\alpha \in L \subseteq \cup\tilde{\gamma}_\alpha$. It does exist, for otherwise $\cap\mathcal{F}_\alpha \setminus \cup\tilde{\gamma}_\alpha \neq \emptyset$, where $\mathcal{F}_\alpha = \{K \in \mathcal{F} : x_\alpha \in K\}$. The latter, however, contradicts the separating property of $\mathcal{F}$. Since λ is a regular cardinal and $|\mathcal{F}| \leq \tau < \lambda$, one can find a set $A_0 \subseteq \lambda$, $|A_0| = \lambda$, and an element $K^* \in \mathcal{F}$ such that $x_\alpha \in K^* \subseteq \cup\tilde{\gamma}_\alpha$ for all $\alpha \in A_0$. Using compactness of K^*, we can assume that $|\tilde{\gamma}_\alpha| < \aleph_0$ for each $\alpha \in A_0$. Thus γ_α is a finite cover of the set $T = \{x_\beta : \beta \in A_0\}$ for each $\alpha \in A_0$, and Lemma 2.3 implies the existence of a subset $A \subseteq A_0$ with $|A| = \lambda$ such that $St(x_\alpha, \gamma_\beta) \cap St(x_\beta, \gamma_\alpha) \neq \emptyset$ for all $\alpha, \beta \in A$.

Case 2. There exists a cardinal k such that $\tau < k < \lambda \leq 2^k$.

The argument here is just the same as in Case 1 with k playing the rôle of τ.

Case 3. $2^k < \lambda$ for each $k < \lambda$ (i.e. λ is a strong limit cardinal). This is the only case to care about thoroughly. Since X is completely regular, we can assume that for each $\alpha < \lambda$ there exists a continuous function $\varphi_\alpha : X \to [0, 1]$ such that $O_\alpha = \varphi_\alpha^{-1}(0, 1]$. Consider continuous mappings $\psi_\alpha = \varphi_\alpha \circ F$ and $\tilde{\psi}_\alpha = \psi_\alpha|_{\sigma^3}$; $\alpha < \lambda$. Note that for every finite $B \subseteq J$ the space Π_B satisfies the inequality $l(\Pi_B) \leq Nag(\Pi_B) \leq \tau$, and hence $l(\sigma) \leq \tau$ by a theorem of [6]. By the same reason we have $l(\sigma^3) \le \tau$. Since $\tilde{\psi}_\alpha$ is a continuous mapping of σ^3 to a second-countable space, $\tilde{\psi}_\alpha$ depends on at most τ coordinates by Corollary 1 of [10] (another way is to apply a theorem of Engelking [3]). Let M_α be the set of all the indices of J the function $\tilde{\psi}_\alpha$ depends on; $|M_\alpha| \le \tau$. Note that in fact we should speak about coordinates of the index set J^3, but 'spilling' the triples of J^3 we get the sets M_α lying in J. Obviously, ψ_α does not depend on $J \setminus M_\alpha$ either, because σ^3 is dense in Z^3. For every $\alpha < \lambda$ denote $N_\alpha = M_\alpha \cup supp(x_\alpha)$ and consider the family $\xi = \{N_\alpha : \alpha < \lambda\}$. If $|\xi| < \lambda$, by the regularity of λ there exists a set $E \subseteq \lambda$ of cardinality λ such that $N_\alpha = N_\beta = B$ for all $\alpha, \beta \in E$. Then $|B| \le \tau$ and we will get Case 1 if we consider the subspace σ_B of Z and the family $\{\gamma_\alpha : \alpha \in E\}$ of covers of σ_B. So it remains to assume that $|\xi| = \lambda$. Then by Lemma 2.2 the family ξ contains a quasidisjoint subfamily η with $|\eta| = \lambda$. Let $\eta = \{N_\alpha : \alpha \in C\}$, where $C \subseteq \lambda$, and suppose that R is a root of η. Diminishing η, we can also assume that $N_\alpha \ne N_\beta$ for all distinct α, β of C. For each $\alpha \in C$ put $N^*_\alpha = N_\alpha \setminus R$. Then $\{N^*_\alpha : \alpha \in C\}$ is a disjoint family of cardinality λ.

We claim that $U = \sigma \cap \pi^{-1}_{N_\alpha} \pi_{N_\alpha}(U)$ for all $U \in \gamma_\alpha$ and $\alpha < \lambda$. First, note that the restriction of π_{N_α} to σ is an open mapping onto $\pi_{N_\alpha}(\sigma)$; therefore by the maximality of a set U ($= U_\alpha(x)$ for some point $x \in \sigma$) it suffices to verify that the open set $\sigma \cap \pi^{-1}_{N_\alpha} \pi_{N_\alpha}(U)$ satisfies the above condition (1). Indeed, take $y, z, y', z' \in \sigma$ such that $y, z \in U_\alpha(x)$, $\pi_{N_\alpha}(y') = \pi_{N_\alpha}(y)$ and $\pi_{N_\alpha}(z') = \pi_{N_\alpha}(z)$. Then $\pi^3_{N_\alpha}(x_\alpha, y', z') = \pi^3_{N_\alpha}(x_\alpha, y, z)$, and the definition of the subset M_α of N_α implies

$$\psi_\alpha(x_\alpha, y', z') = \varphi_\alpha F(x_\alpha, y', z') = \varphi_\alpha F(x_\alpha, y, z) = \psi_\alpha(x_\alpha, y, z).$$

By the definition of $U_\alpha(x)$ we have

$$y, z \in U_\alpha(x) \Longrightarrow \varphi_\alpha F(x_\alpha, y, z) > 0 \Longrightarrow \varphi_\alpha F(x_\alpha, y', z') > 0.$$

Analogously,

$$y, z \in U_\alpha(x) \Longrightarrow \varphi_\alpha F(z, y, x_\alpha) > 0 \Longrightarrow \varphi_\alpha F(z', y', x_\alpha) > 0.$$

Since $O_\alpha = \varphi_\alpha^{-1}(0,1]$, the inequalities above imply that the set $\tilde{U}_\alpha(x) = \sigma \cap \pi^{-1}_{N_\alpha} \pi_{N_\alpha}(U_\alpha(x))$ satisfies (1), and hence $\tilde{U}_\alpha(x) = U_\alpha(x)$.

Denote $\Pi^*_R = \Pi_R \times \{\pi_{J\setminus R}(p^*)\}$ and $\Phi = cl_{\beta\Pi}\Pi^*_R$. Since $|R| \le \tau$, the subspace $\sigma_R = \sigma \cap \Pi^*_R$ of Π satisfies $Nag(\sigma_R) \le \tau$. The set σ_R is dense in Φ, and hence there exists a family $\mathcal{F}$ of closed sets in Φ separating points of σ_R from the points of $\Phi \setminus \sigma_R$, $|\mathcal{F}| \le \tau$. For each $\alpha \in C$, define a point $x^*_\alpha \in \sigma_R$ by $\pi_R(x^*_\alpha) = \pi_R(x_\alpha)$ and $\pi_{J\setminus R}(x^*_\alpha) = \pi_{J\setminus R}(p^*)$. The same argument as in Case 1 gives us an element $K^* \in \mathcal{F}$ and a subset D of C such that $|D| = \lambda$ and $x^*_\alpha \in K^* \subseteq \cup\tilde{\gamma}_\alpha$ for all $\alpha \in D$.

Claim. If $\mu \subseteq \gamma_\alpha$ for some $\alpha \in D$ and $K^* \subseteq \cup\mu$, then $\{x_\beta : \beta \in D\} \setminus \{x_\alpha\} \subseteq \cup\mu$.

Indeed, let $\alpha \in D$ and $\mu \subseteq \gamma_\alpha$, $F \subseteq \cup\mu$. For an arbitrary $\beta \in D \setminus \{\alpha\}$ choose $U \in \mu$ with $x^*_\beta \in U$. The points x_β and x^*_β coincide on all coordinates of $J \setminus N^*_\beta$, and $N^*_\beta \cap N_\alpha = \emptyset$. Since $x^*_\beta \in U$ and $U = \pi^{-1}_{N_\alpha}\pi_{N_\alpha}(U)$, we conclude that $x_\beta \in U$. This proves the claim.

For each $\alpha \in D$, find a finite subfamily μ_α of γ_α that covers the compact set $K^* \cup \{x_\alpha\}$ and apply Lemma 2.4 to the family $\{\mu_\alpha : \alpha \in D\}$ of covers of the sequence $\{x_\alpha : \alpha \in D\}$ (Claim works here). This gives us a subset $A \subseteq D$ of cardinality λ such that $St(x_\alpha, \mu_\beta) \cap St(x_\beta, \mu_\alpha) \neq \emptyset$ for all $\alpha, \beta \in A$. Since $\mu_\alpha \subseteq \gamma_\alpha$, we conclude that

$(*)$ $St(x_\alpha, \gamma_\beta) \cap St(x_\beta, \gamma_\alpha) \neq \emptyset$ for all $\alpha, \beta \in A$.

*Final step.*In each of Cases 1–3 we have found a subset $A \subseteq \lambda$ of cardinality λ with the above property $(*)$. It remains to show that $O_\alpha \cap O_\beta \neq \emptyset$ for all $\alpha, \beta \in A$. To this end apply an argument similar to the proof of Proposition 1 of [16]. Let $\alpha, \beta \in A$ and $\alpha \neq \beta$. By $(*)$ one can find $U \in \gamma_\alpha$, $V \in \gamma_\beta$ and a point $z \in \sigma$ such that $\{x_\alpha, z\} \subseteq V$ and $\{x_\beta, z\} \subseteq U$. Then by the definition of γ_α and γ_β, we have $F(x_\alpha, z, x_\beta) \in O_\alpha \cap O_\beta \neq \emptyset$. □

The following result was earlier announced by the author in [12] for $\tau = \aleph_0$.

2.4 Theorem. *Let $\Pi = \prod_{i \in J} P_i$ be a product of spaces satisfying $Nag(P_i) \leq \tau$ for all $i \in J$, and suppose that $f : \Pi \to X$ is an M-mapping onto a Tikhonov space X. Then every regular cardinal $\lambda > \tau$ is a weak precaliber for X and $cel_\tau X \leq \tau$.*

Proof: It suffices to put $Z = \Pi$ in Theorem 2.1. The inequality $cel_\tau \leq \tau$ follows from [13, Theorem 2.2]. □

2.5 Remark. The subspace Z of the product Π in Theorem 2.1 could not be an *arbitrary* dense set in Π even if the index set J is one-point, i.e. $Nag(\Pi) \leq \tau$. In fact, *every* space is an image of some dense subspace of a *compact space* under an M-mapping. Indeed, for a given space X let X_d be a discrete group with the underlying set X. Then the identity mapping $id_X : X_d \to X$ is obviously an M-mapping and X_d is dense in the compact space βX_d.

2.6 Remark. A proof of the 'weak precaliber' part of Theorem 2.4 can be essentially simplified in comparison with the proof of Theorem 2.1. Indeed, let us have defined the points $x_\alpha \in \Pi$ with $f(x_\alpha) \in O_\alpha$ for all $\alpha < \lambda$. For every $\alpha < \lambda$ there exists a standard open set $V_\alpha \ni x_\alpha$ in Π such that $f(V_\alpha) \subseteq O_\alpha$. Then $V_\alpha = \pi^{-1}_{D(\alpha)}\pi_{D(\alpha)}V_\alpha$ for some finite subset $D(\alpha)$ of J. By the usual $\triangle$-lemma, the family $\xi = \{D(\alpha) : \alpha < \lambda\}$ contains a quasidisjoint subfamily of the same cardinality λ, so we can assume that ξ is quasidisjoint itself and has a root R, $|R| < \aleph_0$. Put $E(\alpha) = D(\alpha) \setminus R$ for each $\alpha < \lambda$. The family $\{E(\alpha) : \alpha < \lambda\}$ is disjoint, and one can find a point $q \in \Pi_{J\setminus R}$ such that $q|_{E(\alpha)} = x_\alpha|_{E(\alpha)}$ for all $\alpha < \lambda$. Denote $\Pi^*_R = \Pi_R \times \{q\}$. Then $V_\alpha \bigcap \Pi^*_R \neq \emptyset$, and hence $f(\Pi^*_R) \bigcap O_\alpha \neq \emptyset$

for all $\alpha < \lambda$. Since $Nag(\Pi_R^*) = Nag(\Pi_R) \leq \tau$, it remains to apply the argument of Case 1 and Final step of the corresponding proof.

2.7 Corollary. *If an M-space X is a continuous image of a product of Lindelöf Σ-spaces, then every regular uncountable cardinal is a weak precaliber for X and X is $\aleph_0$-cellular.*

PROOF: Every continuous mapping onto an M-space is an M-mapping. The conclusion now follows from Theorem 2.4. □

2.8 Corollary. *If X is a dense subspace of a Lindelöf Σ-group, then every regular uncountable cardinal is a weak precaliber for X.*

PROOF: The 'weak precaliber' property is hereditary with respect to dense subspaces. It remains to note that a topological group is an M-space and apply Corollary 2.7. □

It is known that the cellularity can be raised by multiplying of two spaces [14]. Recently Todorčević [15] constructed (in ZFC only) an example of a topological group H with $c(H \times H) > c(H)$. However, the latter is impossible in the class of subgroups of Lindelöf Σ-groups.

2.9 Proposition. *Let G be a subgroup of a Lindelöf Σ-group. Then $c(G \times H) = c(H)$ for every infinite topological group H.*

PROOF: There exists a Lindelöf Σ-group $\hat{G}$ containing G as a subgroup. Then $K = cl_{\hat{G}} G$ is a closed subgroup of $\hat{G}$, and hence is a Lindelöf Σ-group. Since G is dense in K, every uncountable regular cardinal is a weak precaliber for G by Corollary 2.8. In particular, this is the case for τ^+ where $\tau = c(H)$. For the following argument, topological group structures of G and H are unessential. Let γ be a family of open sets in $G \times H$, $|\gamma| = \tau^+$. It suffices to find two distinct elements of γ with a non-empty intersection. We can assume that each element of γ has the form $U \times V$ for some open sets $U \subseteq X$ and $V \subseteq Y$. For every subfamily μ of γ denote $\widetilde{\mu} = \{p_1(O) : O \in \mu\}$, where p_1 is the projection of $G \times H$ onto G. Since τ^+ is a weak precaliber for G, one can find a subfamily μ of γ such that $|\mu| = \tau^+$ and $U \cap U' \neq \emptyset$ for all $U, U' \in \tilde{\mu}$. The inequality $c(H) \leq \tau$ implies that $p_2(O) \cap p_2(O') \neq \emptyset$ for some distinct $O, O' \in \mu$ where p_2 is the projection of $G \times H$ onto H. Since O and O' have rectangular form, we conclude that $O \cap O' \neq \emptyset$. □

By Theorem 1.1 of [9], every topological group G with $l(G) \leq \tau$ is 2^τ-cellular, i.e. $cel_\tau(G) \leq 2^\tau$. Roughly speaking, this result was proved by making use of a 'good' lattice of open mappings of G onto quotient groups of countable pseudocharacter. Again, we show that the existence of an appropriate M-mapping is responsible for this phenomenon.

2.10 Theorem. *Let $f : \Pi \to X$ be an M-mapping of a space Π with $l(\Pi) \leq \tau$ onto X. Then $cel_\tau(X) \leq 2^\tau$.*

PROOF: Suppose the contrary. Then there exists a sequence $\{(K_\alpha, O_\alpha) : \alpha < \lambda\}$ such that $K_\alpha \subseteq O_\alpha \subseteq X$, K_α is a non-empty closed G_τ-set in X, O_α is open in

X and $K_\beta \cap O_\alpha = \emptyset$ whenever $\beta < \alpha < \lambda$; $\lambda = (2^\tau)^+$. Diminishing K_α and O_α if necessary, we can assume that for each $\alpha < \lambda$ there exists a continuous mapping φ_α of X onto a space X_α of weight at most τ such that $K_\alpha = \varphi_\alpha^{-1}\varphi_\alpha K_\alpha$.

Let $F : \Pi^3 \to X$ be a continuous mapping with the property $F(x, y, y) = F(y, y, x) = f(x)$ for all $x, y \in \Pi$. For every $\alpha < \lambda$ pick a point $x_\alpha \in \Pi$ with $f(x_\alpha) \in K_\alpha$ and define a continuous mapping $\psi_{\alpha,\beta} : \Pi \to X_\alpha$ by $\psi_{\alpha,\beta}(x) = \varphi_\alpha F(x_\alpha, x_\beta, x)$ for all $\alpha, \beta < \lambda$ and $x \in \Pi$. Note that $\psi_{\beta,\beta} = \varphi_\beta \circ f$ for all $\beta < \lambda$. Denote $T = \{x_\alpha : \alpha < \lambda\}$ and $T_\alpha = \{x_\beta : \beta < \alpha\}$; $\alpha < \lambda$. If $g : \Pi \to Y$ is an arbitrary mapping to a space Y with $w(Y) \le \tau$, then $|Y| \le 2^\tau$; therefore there exists an ordinal $\delta(g) < \lambda$ such that $g(T_{\delta(g)}) = g(T)$.

The following step is a transfinite construction on $\alpha < \lambda$. Put $\nu(0) = 0$. Suppose that for some $\alpha < \tau^+$ we have already defined ordinals $\nu(\beta) < \lambda$ for all $\beta < \alpha$. If α is limit, we put $\nu(\alpha) = sup_{\beta<\alpha}\nu(\beta)$. Otherwise $\alpha = \beta + 1$ for some β and we put $\mathcal{G}_\alpha = \{\psi_{\gamma,\gamma'} : \gamma, \gamma' \le \beta\}$ and denote by $\mathcal{H}_\alpha$ the family $\{\triangle\Psi : \Psi \subseteq \mathcal{G}_\alpha, |\Psi| \le \tau\}$, where $\triangle\Psi$ is the diagonal product of mappings of Ψ. Obviously, $|\mathcal{H}_\alpha| \le 2^\tau$ and the weight of the space $g(\Pi)$ does not exceed τ for each $g \in \mathcal{H}_\alpha$. Therefore we can define $\nu(\alpha)$ as the maximum of the ordinals $\nu(\beta)$ and $sup\{\delta(g) : g \in \mathcal{H}_\alpha\}$. Clearly, $\nu(\alpha) < \lambda$. This completes our construction.

Put $\nu = sup_{\alpha<\tau^+}\nu(\alpha)$, $\mathcal{G} = \{\psi_{\gamma,\gamma'} : \gamma, \gamma' < \nu\}$ and $h = \triangle\mathcal{G}$. The crucial statement is that the set $C = cl_\Pi T_\nu$ has the property $h(T) \subseteq h(C)$. Indeed, by the construction, for every subfamily Ψ of $\mathcal{G}$ with $|\Psi| \le \tau$ we have $(\triangle\Psi)(T) \subseteq (\triangle\Psi)(T_\nu)$, and the statement follows from the fact that $l(C) \le \tau$.

Choose a point $z \in C$ with $h(z) = h(x_\nu)$. We have $F(z, z, x_\nu) = f(x_\nu) \in K_\nu \subseteq O_\nu$, and the continuity of F in the first argument implies that there exists an open neighbourhood V of z in Π such that $F(x, z, x_\nu) \in O_\nu$ for all $x \in V$. Since z is in the closure of T_ν, one can find $\alpha < \nu$ with $x_\alpha \in V$. Thus we have $F(x_\alpha, z, x_\nu) \in O_\nu$. To get a contradiction it remains to show that $F(x_\alpha, z, x_\nu) \in K_\alpha$.

Since $\psi_{\alpha,\beta} \in \mathcal{G}$ for all $\beta < \nu$, from the choice of the point z it follows that $\varphi_\alpha F(x_\alpha, x_\beta, x_\nu) = \varphi_\alpha F(x_\alpha, x_\beta, z)$ for each $\beta < \nu$. Using the continuity of F in the first argument, we get

$$\varphi_\alpha F(x_\alpha, z, x_\nu) = \varphi_\alpha F(x_\alpha, z, z) = \varphi_\alpha f(x_\alpha) \in \varphi_\alpha(K_\alpha).$$

However, $K_\alpha = \varphi_\alpha^{-1}\varphi_\alpha(K_\alpha)$, whence it follows that the point $p = F(x_\alpha, z, x_\nu)$ belongs to K_α. Thus $p \in O_\nu \cap K_\alpha \ne \emptyset$ and $\alpha < \nu$, a contradiction with the choice of K_α and O_ν. □

2.11 Remark. An easy examination of the above proof shows that we have not used the continuity of the mapping $F : \Pi^3 \to X$ in all its power. In fact, just the separate continuity of F in its arguments is necessary. This enables to consider groups with a separately continuous multiplication (and continuous inverse); let us call them quasitopological. Thus we have the following.

2.12 Corollary. *A quasitopological group G with $l(G) \le \tau$ satisfies $cel_\tau(G) \le 2^\tau$.*

This corollary of Theorem 2.10 is a slight generalization of [9, Theorem 1.1]. Note that one cannot improve Corollary 2.12 by showing $cel_\tau(G) \leq \tau$ even for a topological group G. Indeed, if X is a one-point τ-Lindelöfication of a discrete set of cardinality greater than τ, then the free Abelian group $G = A(X)$ satisfies $l(G) \leq \tau$ and $c(G) > \tau$ (see [11] for details).

2.13 Problem. Does Theorem 2.10 remain valid for a weakly τ-Lindelöf space Π (i.e. for a space each open cover of which contains a subfamily of cardinality $\leq \tau$ with a dense union)?

The following problem is closely connected with the previous one.

2.14 Problem. Let S be a weakly Lindelöf subspace of a product $\prod_{i\in J} P_i$ and suppose that $f : S \to X$ is an M-mapping to a second-countable space X. Does f depend on countably many coordinates?

The answer to Problem 2.14 is "yes" if S is a *subgroup* of a product of *topological groups* P_i, or even if S is an *M-subspace* of a product of *M-spaces* P_i [4, Theorem 1.1].

Combining the argument exposed in the proof of Theorems 2.2 and 2.10, we can prove a result generalizing Proposition 6 of [18]. This requires the following notation. Suppose $f : \Pi \to Y$ and $g : \Pi \to Z$ are continuous mappings and f is onto. We write $f \prec g$ if there exists a continuous mapping $h : Y \to Z$ such that $g = h \circ f$.

2.15 Theorem. *An image of a $\Sigma(\aleph_0)$-space under an M-mapping is $\aleph_0$-cellular.*

Proof: Let $f : \Pi \to X$ be an M-mapping of a $\Sigma(\aleph_0)$-space Π onto X and suppose that $F : \Pi^3 \to X$ witnesses that. If X is not $\aleph_0$-cellular, there exists a sequence $\{(K_\alpha, O_\alpha) : \alpha < \omega_1\}$ such that $K_\alpha \subseteq O_\alpha \subseteq X$, K_α is a non-empty G_δ-set in X, O_α is open and $K_\beta \cap O_\alpha = \emptyset$ whenever $\beta < \alpha < \omega_1$. For each $\alpha < \omega_1$ pick a point $x_\alpha \in \Pi$ with $f(x_\alpha) \in K_\alpha$. We can also assume that for each $\alpha < \omega_1$ there exists a continuous function $\varphi_\alpha : X \to [0,1]$ such that $K_\alpha = \varphi_\alpha^{-1}(1)$ and $X \setminus O_\alpha = \varphi_\alpha^{-1}(0)$. Define a continuous mapping $\psi_{\alpha,\beta} : \Pi \to [0,1]$ by $\psi_{\alpha,\beta}(x) = \varphi_\alpha F(x_\alpha, x_\beta, x)$ for all $\alpha, \beta < \omega_1$ and $x \in \Pi$. Let two covers $\mathcal{K}$ and $\mathcal{C}$ of Π witness that Π is a $\Sigma(\aleph_0)$-space, $|\mathcal{K}| \leq \aleph_0$. The family $\mathcal{K}$ can be chosen closed under finite intersections.

Let α_0 be a countable ordinal. Put $g_0 = \triangle\{\psi_{\beta,\gamma} : \beta, \gamma \leq \alpha_0\}$. Then the space $Y_0 = g_0(\Pi)$ is second-countable. Suppose that for some integer n we have defined an ordinal $\alpha_n < \omega_1$ and a continuous mapping $g_n : \Pi \to Y_n$ onto a space Y_n with $w(Y_n) \leq \aleph_0$. Since Y_n is hereditarily separable, for each $K \in \mathcal{K}$ there exists an ordinal $\delta = \delta_n(K) < \omega_1$ such that $g_n(K \cap T_\delta)$ is dense in $g_n(K \cap T)$, where $T = \{x_\alpha : \alpha < \omega_1\}$ and $T_\delta = \{x_\nu : \nu < \delta\}$. Put $\alpha_{n+1} = max\{\alpha_n, sup\{\delta_n(K) : K \in \mathcal{K}\}\}$ and $g_{n+1} = g_n \triangle(\triangle\{\psi_{\beta,\gamma} : \beta, \gamma \leq \alpha_{n+1}\})$; the symbol $\triangle$ stands for the diagonal product of mappings. Thus we have defined an increasing sequence $\{\alpha_n : n \in \omega\}$ of countable ordinals and can now put $\alpha = sup_{n\in\omega}\alpha_n$ and $g = \triangle\{\psi_{\beta,\gamma} : \beta, \gamma < \alpha\}$. Then $\alpha < \omega_1$ and the space $Y = g(\Pi)$ is second-countable.

From the construction it follows that the following conditions are fulfilled:

(1) $g(K \cap T_\alpha)$ is dense in $g(K \cap T)$ for each $K \in \mathcal{K}$;

(2) $g \prec \psi_{\beta,\gamma}$ for all $\beta, \gamma < \alpha$.

We claim that $g(x_\alpha) \in g(cl_\Pi T_\alpha)$. Indeed, choose $C^* \in \mathcal{C}$ with $x_\alpha \in C^*$. Denote by $\mathcal{B}$ a countable base of the space Y at $g(x_\alpha)$. By (1), the family $\mathcal{F} = \{g^{-1}(cl_Y U) \cap K \cap cl_\Pi T_\alpha : U \in \mathcal{B}, C^* \subseteq K \in \mathcal{K}\}$ consists of non-empty closed subsets of Π. The set $g^{-1}(cl_Y U) \cap cl_\Pi T_\alpha$ meets C^* for each $U \in \mathcal{B}$, otherwise by the choice of $\mathcal{K}$ there exists $K^* \in \mathcal{K}$ such that $C^* \subseteq K^* \subseteq \Pi \setminus (g^{-1}(cl_Y U) \cap cl_\Pi T_\alpha)$, which contradicts the fact that all elements of $\mathcal{F}$ are non-empty. Thus, the countable family $\mathcal{F}^* = \{g^{-1}(cl_Y U) \cap C^* \cap cl_\Pi T_\alpha : U \in \mathcal{B}\}$ consists of non-empty closed subsets of C^* and by the choice of $\mathcal{B}$ has the finite intersection property. Since C^* is countably compact, we have $\emptyset \neq \cap \mathcal{F}^* = g^{-1} g(x_\alpha) \cap C^* \cap cl_\Pi T_\alpha$. Consequently, $g(x_\alpha) \in g(cl_\Pi T_\alpha)$.

Pick a point $z \in cl_\Pi T_\alpha$ with $g(z) = g(x_\alpha)$. Since $F(z, z, x_\alpha) = f(x_\alpha) \in K_\alpha \subseteq O_\alpha$, there exists a neighbourhood V of z such that $F(V \times \{z\} \times \{x_\alpha\}) \subseteq O_\alpha$. From $z \in cl_\Pi T_\alpha$ it follows that $x_\beta \in V$ for some $\beta < \alpha$, and we have $F(x_\beta, z, x_\alpha) \in O_\alpha$. Since $g(x_\alpha) = g(z)$, (2) implies that $\psi_{\beta,\gamma}(x_\alpha) = \psi_{\beta,\gamma}(z)$, i.e. $\varphi_\beta F(x_\beta, x_\gamma, x_\alpha) = \varphi_\beta F(x_\beta, x_\gamma, z)$ for all $\gamma < \alpha$. Taking into account the continuity of F in the first argument and the fact that $z \in cl_\Pi T_\alpha$, we get the equalities

$$\varphi_\beta F(x_\beta, z, x_\alpha) = \varphi_\beta F(x_\beta, z, z) = \varphi_\beta f(x_\beta) = 1.$$

Since $K_\beta = \varphi_\beta^{-1}(1)$, the latter means that $y = F(x_\beta, z, x_\alpha) \in K_\beta$. Thus $y \in K_\beta \cap O_\alpha \neq \emptyset$ and $\beta < \alpha$, which contradicts the choice of the sequence $\{(K_\nu, O_\nu) : \nu < \omega_1\}$. □

Note again that the separate continuity of the mapping $F : \Pi^3 \to X$ was only used in the proof of the above theorem. Thus we have the following corollary.

2.16 Corollary. *If a quasitopological group G is a continuous image of some $\Sigma(\aleph_0)$-space, then G is $\aleph_0$-cellular.*

2.17 Corollary. *Suppose a space Π admits a quasiperfect (i.e. continuous closed with countably compact fibers) mapping onto a space with a countable network. Then every image of Π under an M-mapping is $\aleph_0$-cellular.*

Proof: Let $h : \Pi \to Z$ be a quasiperfect mapping of Π onto a space Z with a countable network $\mathcal{N}$. Since Z is regular, we can assume that $\mathcal{N}$ consists of closed sets. Put $\mathcal{C} = \{h^{-1}(z) : z \in Z\}$ and $\mathcal{K} = \{h^{-1}(N) : N \in \mathcal{N}\}$. Then the covers $\mathcal{C}$ and $\mathcal{K}$ of Π witness that Π is a $\Sigma(\aleph_0)$-space. It remains to apply Theorem 2.15. □

2.18 Problem. Suppose $f : \Pi \to X$ is an M-mapping of a countably compact space Π onto X. Is then every regular uncountable cardinal a caliber for X?

2.19 Problem (see also [13, Problem 2.4]). Does an image of a pseudocompact space under an M-mapping have the Souslin property?

It is still unknown whether every pseudocompact M-space has the Souslin property; see [18] for details.

We conclude with a little bit alien problem to the area.

2.20 Problem (see [13, Problem 2.5]). If X is an image of a compact space under an M-mapping, must X be dyadic?

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