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Quasitrivial left distributive groupoids

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Abstract. Left distributive quasitrivial groupoids are completely described and those of them which are subdirectly irreducible are found. There are also determined all left distributive algebras $A = A(*, \circ)$ such that $A(*)$ is a quasitrivial groupoid.

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An algebra $A = A(*, \circ)$ with two binary operations $*$ and $\circ$ is said to be a *left distributive algebra* (or an LD-algebra) [LavFr], [DehAd] if

(P1) $$(a \circ b) \circ c = a \circ (b \circ c)$$

(P2) $$(a \circ b) * c = a * (b * c)$$

(P3) $$a \circ b = (a * b) \circ a$$

(P4) $$a * (b \circ c) = (a * b) \circ (a * c)$$

for any $a, b, c \in G$. The left distributive law

$$a * (b * c) = (a * b) * (a * c)$$

is a consequence of identities (P2–3). A groupoid fulfilling this law is called *left distributive* (or an LD-groupoid).

A groupoid $B = B(*)$ is said to be *quasitrivial* if

$$a * b \in \{a, b\}$$

for any $a, b \in B$.

In this paper we determine all quasitrivial LD-groupoids. We also determine all LD-algebras $A(*, \circ)$ such that $A(*)$ is quasitrivial and all subdirectly irreducible quasitrivial LD-groupoids. We show that subdirectly irreducible quasitrivial LD-groupoids form a proper class.

The groupoid $A(*)$ with $a*b = b$ for all $a, b \in A$ will be called *discrete*. Discrete groupoids are quasitrivial and left distributive. (Such groupoids are often called semigroups of left units or semigroups of right zeros.)

Let G be a group and put $a * b = aba^{-1}$ for any $a, b \in G$. Then $G(*, \cdot)$ is an LD-algebra. Suppose that A is a quasitrivial subgroupoid of $G(*)$. Then $ab = ba$

for any $a, b \in A$, and we see that $A(*)$ is discrete. We shall show that there are many quasitrivial LD-groupoids that are not discrete.

Quasitrivial groupoids that are both left and right distributive have been described in [JeKe] by Ježek and Kepka. Kepka has also studied [KepQ] quasitrivial groupoids in the general case of linear identities (i.e. identities in which each variable occurs exactly once at both sides).

Our paper is a modest contribution to the ongoing investigation of left distributive structures. While the deepest results concern free monogenerated LD-groupoids [LavFr], [DehBr], idempotent LD-groupoids have recently received also some attention [DKM]. (A groupoid is *idempotent*, if $a * a = a$ for all $a \in A$. Quasitrivial groupoids are idempotent.)

For each quasitrivial groupoid $A = A(*)$ define relation $\gamma = \gamma_A$ by

$$(a, b) \in \gamma \iff a * b = a.$$

Lemma 1. *Let $A = A(*)$ be a quasitrivial groupoid. Then $a * b = a * (a * b) = (a * b) * b$ for any $a, b \in A$.*

By a *quasiordering* we mean any reflexive and transitive relation. A quasiordering $\leq$ of a set M will be called *downward rectified*, if $a \in M$ and $b \in M$ are comparable whenever there exists $c \in M$ with $a \leq c$ and $b \leq c$ ($a \in M$ and $b \in M$ are said to be *comparable* if $a \leq b$ or $b \leq a$).

Proposition 1. *A quasitrivial groupoid $A(*)$ is left distributive iff γ_A is a downward rectified quasiordering of A.*

Proof: Suppose first that γ is a downward rectified quasiordering. For $a, b, c \in A$ put $l = a * (b * c)$ and $r = (a * b) * (a * c)$.

(i) $(a, b) \in \gamma$ and $(b, c) \in \gamma$. Then $(a, c) \in \gamma$ by transitivity of γ, and hence $l = a = r$.

(ii) $(a, b) \in \gamma$ and $(b, c) \notin \gamma$. Then $l = a * c = a * (a * c) = r$.

(iii) $(a, b) \notin \gamma$ and $(b, c) \in \gamma$. If $(a, c) \notin \gamma$, then $l = b = r$. Since γ is downward rectified, $(a, c) \in \gamma$ implies $(b, a) \in \gamma$, and we have $l = b = r$ again.

(iv) $(a, b) \notin \gamma$ and $(b, c) \notin \gamma$. In this case $l = a * c$ and $r = b * (a * c)$. If $(a, c) \notin \gamma$, then $l = c = r$. If $(a, c) \in \gamma$, then $(b, a) \in \gamma$ implies $(b, c) \in \gamma$ by transitivity of γ. Thus $(b, a) \notin \gamma$ and $l = a = r$.

On the other hand suppose that $A(*)$ is quasitrivial and left distributive. If $(a, b) \in \gamma$ and $(b, c) \in \gamma$, then $a*c = a*(a*c) = (a*b)*(a*c) = a*(b*c) = a*b = a$. The relation γ is therefore transitive. Furthermore, let $(a, c) \in \gamma$, $(b, c) \in \gamma$ and $(a, b) \notin \gamma$. Then $b * a = (a * b) * (a * c) = a * (b * c) = a * b = b$. It follows that γ is downward rectified. □

Let $A_i = A_i(*)$, $i \in I$ be pairwise disjoint left distributive groupoids. Define a groupoid $V = V(A_i; i \in I)$ on $\cup(A_i; i \in I)$ so that

$$a * b = \begin{cases} b & \text{if } a \in A_i,\ b \in A_j \text{ and } i \neq j, \\ a *_i b & \text{if } a, b \in A_i. \end{cases}$$

Lemma 2. *Let A_i, $i \in I$ be pairwise disjoint LD-groupoids. Then $V = V(A_i; i \in I)$ is also an LD-groupoid. If all A_i, $i \in I$ are idempotent (or quasitrivial), then V is idempotent (or quasitrivial), too.*

Proof: Only the left distributivity requires a proof. For $a, b, c \in V$ put $l = a * (b * c)$ and $r = (a * b) * (a * c)$. Suppose that $a \in A_i$, $b \in A_j$ and $c \in A_k$. If $i = j = k$, then $l = r$ by the hypothesis. If i, j, k are pairwise distinct or $i = j \neq k$, then $l = c = r$. If $i \neq j = k$, then $l = b *_j c = r$, and if $i = k \neq j$, then $l = a *_i c = r$. □

Let $A(*)$ be a quasitrivial groupoid and denote by ρ the least equivalence containing γ. The equivalence classes of ρ are called *components* of $A(*)$. A quasitrivial groupoid with only one component is said to be *connected*.

Corollary 1. *If $A = A(*)$ is a quasitrivial LD-groupoid and A_i, $i \in I$ are its components, then $A = V(A_i; i \in I)$.*

Lemma 3. *Let $A(\circ)$ be a semigroup and $A(*)$ a discrete LD-groupoid. Then $A(*, \circ)$ is an LD-algebra iff $A(\circ)$ is commutative.*

Proof: If $A(*, \circ)$ is an LD-algebra, then $a \circ b = (a * b) \circ a = b \circ a$ for any $a, b \in A$. If $A(\circ)$ is commutative, then the axioms of LD-algebras clearly hold. □

Lemma 4. *Let $S(\circ)$ and $T(\circ)$ be disjoint semigroups. Extend $\circ$ to $U = S \cup T$ so that $s \circ t = s = t \circ s$ for any $s \in S$, $t \in T$. Then $U(\circ)$ is a semigroup again.*

Stepping out of our main line, we note:

Proposition 2. *Let $C(*, \circ)$ and $H(*, \circ)$ be disjoint LD-algebras, and suppose that $H(*)$ is discrete. For $A = C \cup H$ define $A(*, \circ)$ so that:*

(i) *$C(*, \circ)$ and $H(*, \circ)$ are subalgebras of $A(*, \circ)$ and*
(ii) *if $c \in C$ and $h \in H$, then $c \circ h = h \circ c = c = h * c$ and $c * h = h$.*

Then $A(*, \circ)$ is an LD-algebra again.

Proof: Fix such $a, b, c \in A$ that $\{a, b, c\} \cap C \neq \emptyset \neq \{a, b, c\} \cap H$. Assume first $a \in H$, then $a \in C$ and $b \in H$, and finally $a, b \in C$ and $c \in H$. In each of these cases, (P2–4) can be verified immediately. (P1) follows from Lemma 4. □

For a quasitrivial LD-groupoid $A = A(*)$ and $a, b \in A$ write $a||_A b$ (or just $a||b$), if a and b are not comparable with respect to γ_A.

Lemma 5. *Let $A(*)$ be a quasitrivial LD-groupoid. Then $*$ is associative iff*

$$(\dagger) \qquad a||b \text{ and } (b, c) \in \gamma \implies b = c$$

holds for any $a, b, c \in A$.

Proof: Let $*$ be associative and suppose that $a||b$ and $(b, c) \in \gamma$ for some $a, b, c \in A$. Then $(a, c) \notin \gamma$ because γ is downward rectified. Hence $b = b * (a * c) = (b * a) * c = c$. On the other hand let $(\dagger)$ be satisfied by all $a, b, c \in A$. Fix

$a, b, c \in A$ and put $l = a * (b * c)$ and $r = (a * b) * c$. Assume first $(a, b) \in \gamma$. If $(b, c) \in \gamma$, then $l = a = r$. If $(b, c) \notin \gamma$, then $l = a * c = r$. Assume now $(a, b) \notin \gamma$. If $(b, c) \in \gamma$, then $l = b = r$. If $(b, c) \notin \gamma$, then $l = a * c$ and $r = c$. Thus only the case $(a, c) \in \gamma$, $(b, c) \notin \gamma$ and $(a, b) \notin \gamma$ need to be considered. Then $(b, a) \notin \gamma$ by the transitivity of γ, and hence (†) provides $a = c$. □

Call an LD-groupoid $A(*)$ *quasilinear*, if it is quasitrivial and $(a, b) \in \gamma$ or $(b, a) \in \gamma$ for any $a, b \in A$.

Lemma 6. *Let $A(*)$ be a quasilinear LD-groupoid. Put $a \circ b = a * b$ for any $a, b \in A$. Then $A(*, \circ)$ is an LD-algebra.*

Proof: Let $a, b \in A$. If $(a, b) \in \gamma$, then $(a * b) * a = a = a * b$, and if $(a, b) \notin \gamma$, then $(b, a) \in \gamma$ and $(a * b) * a = b = a * b$ too. $A(*)$ is associative by Lemma 5 and (P1–4) follow. □

Let $H(\circ)$ be a commutative semigroup and $I \subseteq H$ its ideal. I is said to be *prime*, if $a \circ b \in I$ implies $a \in I$ or $b \in I$ for any $a, b \in H$. The set of all prime ideals will be denoted $\mathcal{P}(H(\circ))$. Note that $\emptyset$ and H belong to $\mathcal{P}(H(\circ))$.

For disjoint LD-algebras $C = C(*, \circ)$ and $H = H(*, \circ)$, $H(\circ)$ commutative, and a mapping $\theta : C \to \mathcal{P}(H(\circ))$, define on $A = C \cup H$ operations $*$ and $\circ$ so that:

(A1) $C(*, \circ)$ and $H(*, \circ)$ are subalgebras of $A(*, \circ)$.

(A2) $h \circ c = c \circ h = c = h * c$ if $h \in H$ and $c \in C$.

(A3) $c * h = c$ if $h \in H$, $c \in C$ and $h \in \theta(c)$.

(A4) $c * h = h$ if $h \in H$, $c \in C$ and $h \notin \theta(c)$.

The algebra $A(*, \circ)$ will be denoted $A(C, H, \theta)$.

Lemma 7. *Let $C = C(*, \circ)$ and $H = H(*, \circ)$ be disjoint LD-algebras. Suppose that $C(*)$ is quasilinear with $a \circ b = a * b$ for all $a, b \in C$ and that $H(*)$ discrete. Furthermore, let $\theta : C \to \mathcal{P}(H(\circ))$ be a mapping such that $\theta(b) \subseteq \theta(a)$ for any $a, b \in C$ with $(a, b) \in \gamma_C$. Then $A(C, H, \theta)$ is an LD-algebra.*

Proof: (P1) holds by Lemma 4. Fix now $a, b, c \in A = H \cup C$ such that $C \cap \{a, b, c\} \neq \emptyset \neq H \cap \{a, b, c\}$. If $a \in H$, then (P2–4) can be verified immediately. Let $a \in C$ and assume $b \in H$. Then $a \circ b = a$ and $(a * b) \circ a$ is $b \circ a = a$ or $a \circ a = a$. This proves (P3). Now $(a \circ b) * c = a * c = a * (b * c)$ and if $c \in C$, then $a * (b \circ c) = a * c = a \circ (a * c)$ by Lemma 1. Thus $a * (b \circ c) = (a * b) \circ (a * c)$ for $c \in C$, and for $c \in H$ we obtain $a * (b \circ c) = b \circ c = (a * b) \circ (a * c)$, if $b \circ c \notin \theta(a)$. If $b \circ c \in \theta(a)$, then $b \in \theta(a)$ or $c \in \theta(a)$, and hence $a * (b \circ c) = a = (a * b) \circ (a * c)$.

Assume $b \in C$ and $c \in H$. Then $a * (b \circ c) = a * b$ and $(a * b) \circ (a * c)$ equals $a * b$ or $(a * b) \circ a$. By Lemma 6 $(a * b) \circ a = a * b$, and hence (P4) is true. Put now $l = (a \circ b) * c = (a * b) * c$ and $r = a * (b * c)$. Assume first $(a, b) \in \gamma$. If $c \notin \theta(a)$, then $c \notin \theta(b) \subseteq \theta(a)$ and $l = c = r$. If $c \in \theta(a)$, then $l = a$ and r is $a * b = a$ or $a * c = a$. For $(a, b) \notin \gamma$ we distinguish the cases $c \in \theta(b)$ and $c \notin \theta(b)$. If $c \in \theta(b)$, then $l = b * c = b = a * b = r$. If $c \notin \theta(b) \supseteq \theta(a)$, then $l = c = r$. □

For a quasitrivial LD-groupoid $A(*)$ define its *core* as the set of all $a \in A$ such that there exists $b \neq a$ with $(a,b) \in \gamma_A$. If C is the core of A, then call its complement $H = A \backslash C$ *hull* of A. There is $h * a = a$ for any $h \in H$ and $a \in A$. For every $c \in C$ denote by H_c the set of all $h \in H$ with $(c,h) \in \gamma_A$.

Lemma 8. *Let $A(*)$ be a quasitrivial LD-groupoid with a core C and a hull H. If $a, b \in C$ and $(a,b) \in \gamma$, then $H_b \subseteq H_a$.*

Proof: If $h \in H_b$, then $(b,h) \in \gamma$, and thus by transitivity $(a,h) \in \gamma$ too. □

Lemma 9. *Let $A(*,\circ)$ be an LD-algebra and suppose that $A(*)$ is quasitrivial, $C \subseteq A$ is its core and $H = A \backslash C$ its hull. Then:*

(i) *$C(*)$ is quasilinear,*
(ii) *$c \circ d = c * d$ for any $c, d \in C$,*
(iii) *$H(\circ)$ is a commutative subsemigroup of $A(\circ)$,*
(iv) *$H_c \in \mathcal{P}(H(\circ))$ for any $c \in C$,*
(v) *$A(*,\circ) = A(C,H,\theta)$, if $\theta(c) = H_c$ for any $c \in C$.*

Proof: The proof is divided into a series of separate steps:

(1) If $(a,b) \in \gamma$ and $a \neq b$, then $a \circ b = a = a * b$.
This follows from $a = a * (b * b) = (a \circ b) * b$.
(2) If $(a,b) \notin \gamma$, then $a \circ b = b \circ a$.
Clearly, $a \circ b = (a * b) \circ a = b \circ a$.
(3) If $(b,c) \in \gamma$, $b \neq c$ and $a||b$, then $a \circ b = b \circ a = b$.
We have $a * (b * c) = b = (a \circ b) * c$. There is $b \neq c$, and so $b = a \circ b$. By (2) $a \circ b = b \circ a$.
(4) $C(*)$ is quasilinear.
Suppose there are $a, b \in C$ with $a||b$. Let $(a,c) \in \gamma$ and $(b,d) \in \gamma$. By (3) $a = a \circ b = b$, a contradiction.
(5) If $a, b \in C$, then $a \circ b = a * b$.
For $a = b$ let $h \in A$ be such that $a \neq h$ and $(a,h) \in \gamma$. By (1) $a = a \circ h = (a * h) \circ a = a \circ a$. Assume $a \neq b$. If $(a,b) \in \gamma$, use (1). If $(a,b) \notin \gamma$, then $(b,a) \in \gamma$ by (4) and $a \circ b = b \circ a = b$ by (2) and (1).
(6) If $b \in C$ and $a \in H$, then $a \circ b = b \circ a = b$.
There exists $c \in A$ with $(b,c) \in \gamma$ and $b \neq c$. If $a||b$, use (3). If $(b,a) \in \gamma$, use (1) and (2).
(7) If $g, h \in H$, then $g \circ h = h \circ g \in H$.
By (2), $g \circ h = h \circ g$. Assume $g \circ h \in C$. Then there exists $c \in A$ with $c \neq g \circ h$ and $(g \circ h, c) \in \gamma$. Then $c = g * (h * c) = (g \circ h) * c = g \circ h$, a contradiction.
(8) $H_c \in \mathcal{P}(H(\circ))$ for any $c \in C$.
Let $h \in H_c$ and $g \in H$. Then $c*(h \circ g) = (c*h) \circ (c*g) = c \circ (c*g)$. However, $(c,g) \in \gamma$ implies $c \circ (c * g) = c$, and $(c,g) \notin \gamma$ implies $c \circ (c * g) = c$, too. H_c is therefore an ideal. Suppose now that $g \circ h \in H_c$ for $g, h \in H$ and neither $g \in H_c$ nor $h \in H_c$. Then $c * (g \circ h) = c \neq g \circ h = (c * g) \circ (c * h)$, a contradiction.

To conclude note that (i) is (4), (ii) is (5), (iii) is (7), (iv) is (8), (A1) follows from (ii) and (iii) and (A2–4) follow from (6) and the definitions of H and H_c. □

If $\leq$ linearly orders a set S, then $\min_{\leq}$ is a commutative associative quasitrivial binary operation and every ideal of $S(\min_{\leq})$ is prime. Combining Lemma 3, Lemma 7, Lemma 8 and Lemma 9 we can thus state:

Proposition 3. *Let $A(*)$ be a quasitrivial LD-groupoid with a core C. A binary operation $\circ$ on A, such that $A(*, \circ)$ is an LD-algebra, can be defined iff $C(*)$ is quasilinear.*

Moreover, if $C(*)$ is quasilinear, then $\circ$ can be always chosen to be quasitrivial, too.

Proposition 4. *Let $A(*)$ be a quasitrivial LD-groupoid with a quasilinear core C and a hull H. If $\circ$ is a commutative associative binary operation on H, and $\theta : C \to \mathcal{P}(H(\circ))$ a mapping such that $\theta(b) \subseteq \theta(a)$ for $a, b \in C$ with $(a, b) \in \gamma$, and if $a \circ b$ is defined to equal $a * b$ for all $a, b \in C$, then $A(C, H, \theta)$ is an LD-algebra. Moreover, all binary operations $\circ$ on A such that $A(*, \circ)$ is an LD-algebra, can be obtained in this way.*

We turn now our attention to the congruences of quasitrivial LD-groupoids. At the beginning we formulate several easy lemmas pertaining to quasitrivial groupoids in general. Fix a quasitrivial groupoid $A = A(*)$. For $B \subseteq A$ denote ε_B the equivalence on A given by $(a, b) \in \varepsilon_B$ iff $\{a, b\} \subseteq B$ or $a = b$. Furthermore, denote (generically) by $\mathcal{E}$ the set of all $B \subseteq A$ such that ε_B is a congruence of $A(*)$, and by $\mathcal{E}_2$ the subset of $\mathcal{E}$ consisting of all $B \in \mathcal{E}$ with $\operatorname{card}(B) \geq 2$. Finally, put $E(A) = \cap(B;\ B \in \mathcal{E}_2)$.

Lemma 10. *Let $A = A(*)$ be a quasitrivial groupoid and σ an equivalence on A. Then σ is a congruence of A if and only if $(a, a') \in \sigma$, $(b, b') \in \sigma$, $(a, b) \notin \sigma$ and $(a, b) \in \gamma$ imply $(a', b') \in \gamma$ for any $a, b, a', b' \in A$.*

Lemma 11. *Let $B \subseteq A$. Then $B \in \mathcal{E}$ if and only if*

$$(a, b) \in \gamma \implies (a, b') \in \gamma \qquad \text{and} \qquad (b, a) \in \gamma \implies (b', a) \in \gamma$$

for any $b, b' \in B$ and $a \in A \backslash B$.

Lemma 12. *If σ is a congruence of $A(*)$ and B is an equivalence class of σ, then $B \in \mathcal{E}$.*

Lemma 13. *$A(*)$ is subdirectly irreducible iff $E(A)$ contains at least two elements or* $\operatorname{card}(A) \leq 1$.

Lemma 14. *If $B \in \mathcal{E}$ intersects at least two different components of $A(*)$, then it can be expressed as a union of components of $A(*)$. On the other hand, every union of components of $A(*)$ belongs to $\mathcal{E}$.*

Lemma 15. *A disconnected quasitrivial groupoid $A(*)$ is subdirectly irreducible iff it contains exactly two components, one of them subdirectly irreducible and the other one consisting of just one element. If A contains more than two elements and is disconnected and subdirectly irreducible, and if B is its non-trivial component, then $E(A) = E(B)$.*

From here on assume that $A(*)$ is a quasitrivial LD-groupoid and denote by η the kernel of the quasiordering γ; i.e. $(a,b) \in \eta$ iff $(a,b) \in \gamma$ and $(b,a) \in \gamma$. Note that γ is an ordering of A iff $\eta = \mathrm{id}_A$.

From Lemma 10, Lemma 11 and from the transitivity of γ one obtains:

Lemma 16.

(i) *η is a congruence of $A(*)$.*
(ii) *If D is an equivalence class of η and $B \subseteq D$, then $B \in \mathcal{E}$.*
(iii) *If η contains a class with at least three elements, then $E = \emptyset$.*
(iv) *If η contains at least two distinct classes D_1, D_2 with $\mathrm{card}(D_i) \geq 2$, $1 \leq i \leq 2$, then $E = \emptyset$.*
(v) *If η contains a class with at least two elements, then $A(*)$ is simple iff $\mathrm{card}(A) = 2$.*

For every $a \in A$ denote by $[\,a\,]$ the set $\{b \in A;\ (a,b) \in \gamma\}$.

Lemma 17. *$[\,a\,] \in \mathcal{E}$ for every $a \in A$.*

Proof: Let $(a,b) \in \gamma$, $(a,b') \in \gamma$ and $(a,c) \notin \gamma$. Then $(b,c) \notin \gamma$ and from $(c,b) \in \gamma$ we deduce that c and a must be comparable with respect to γ. Thus $(c,a) \in \gamma$ and $(c,b') \in \gamma$ by transitivity. By Lemma 11 $[\,a\,]$ belongs to $\mathcal{E}$. □

A quasitrivial LD-groupoid $A(*)$ will be called *linear*, if γ_A is a linear ordering (i.e. $A(*)$ is quasilinear and $\eta = \mathrm{id}_A$).

Lemma 18. *If the core of $A(*)$ is not linear and η is id_A, then $E(A)$ is $\emptyset$.*

Proof: By our hypothesis there can be found incomparable elements a and b in the core of $A(*)$. Both $[\,a\,]$ and $[\,b\,]$ belong to $\mathcal{E}_2$ and $[\,a\,] \cap [\,b\,] = \emptyset$. □

A subset Q of a linearly ordered set $(P, \leq)$ will be called *downward dense* (in P), if $\emptyset \neq Q \cap \{x \in P;\ a \leq x < b\}$ for any $a, b \in P$, $a < b$.

For an LD-groupoid $A(*)$ with a core C put $\overline{C} = \{B \subseteq C;\ B = \{b \in C;\ (b,e) \in \gamma\}$ for some $e \in A\}$, order $\overline{C}$ by inclusion, denote the ordering of $\overline{C}$ by $\overline{\gamma}$, and assume that $\eta = \mathrm{id}_C$. Then $c \to \{b \in C;\ (b,c) \in \gamma\}$ embeds (C,γ) into $(\overline{C},\overline{\gamma})$. Using this embedding, identify C with a subset of $\overline{C}$. Let H be the hull of $A(*)$. We extend $\overline{\gamma}$ to $\overline{C} \cup H$ in the following way: If $\{a,b\} \subseteq H \cup \overline{C}$ intersects H, then $(a,b) \in \overline{\gamma}$ iff either $a = b$, or $a \in \overline{C}$, $b \in H$ and $(c,b) \in \gamma$ for any $c \in C$ with $(c,a) \in \overline{\gamma}$. Then $\overline{\gamma}$ is an ordering of $\overline{C} \cup H$ and $\gamma = \overline{\gamma} \cap (A \times A)$. By the definition of $\overline{C}$, for any $h \in H$ there exists $\sup_{\overline{\gamma}}\{c \in \overline{C};\ (c,h) \in \overline{\gamma}\}$ and this supremum is in $\overline{C}$. For any $a \in \overline{C}$ denote $\mathrm{card}\{h \in H;\ a = \sup_{\overline{\gamma}}\{c \in \overline{C};\ (c,h) \in \overline{\gamma}\}\}$ by $\deg(a)$. Note that $\deg(a) = 0$ implies $a \in C$ for any $a \in \overline{C}$. If $B \subseteq C$, then denote by B'

the set $\{c \in \overline{C};\ (c,b) \in \overline{\gamma}$ for some $b \in B\}$. If $s = \sup_{\overline{\gamma}} B$ exists and $s \neq \sup_{\overline{\gamma}} C$, put $\overline{B} = B' \cup \{s\}$, otherwise define $\overline{B}$ as B'.

Proposition 5. *Let $A = A(*)$ be a connected quasitrivial LD-groupoid with a core C and a hull H, and assume that $\eta = \mathrm{id}_A$. Put $S = \{h \in H;\ (a,h) \in \gamma$ for all $a \in C\}$, $M = \{c \in C;\ (a,c) \in \gamma$ for all $a \in C\}$ and $C^* = C \backslash M$. Then:*

- (i) *If C is linear, $\mathrm{card}(S) = 2$, $\deg(c) \leq 1$ for all $c \in \overline{C^*}$, and if the set $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C}$, then $E(A) = S$.*
- (ii) *If C is linear, $\mathrm{card}(S) \leq 1$, $\deg(c) \leq 1$ for all $c \in \overline{C^*}$, and if the set $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C}$, then $E(A) = S \cup M$.*
- (iii) *If C is linear, $\mathrm{card}(S) = 1$, $\deg(c) \leq 1$ for all $c \in \overline{C^*}$, and if the set $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C^*}$ and there exists $m \in C^*$ with $\deg(m) = 0$ and $(c,m) \in \gamma$ for all $c \in C^*$, then $E(A) = M$.*
- (iv) *$E(A) = \emptyset$ in all other cases.*

In particular, $\mathrm{card}(E(A)) \leq 2$.

Proof: Assume that $E(A) \neq \emptyset$. We shall show that then one of the cases (i)–(iii) applies and, in parallel, we shall compute $E(A)$ in these cases.

C is linear by Lemma 18. Moreover, by Lemma 11 every subset of S belongs to $\mathcal{E}$, and thus $\mathrm{card}(S) \leq 2$. As $\mathrm{card}([\,c\,]) \geq 2$ for every $c \in C$, $E(A)$ is contained in $\cap([\,c\,];\ c \in C) = S \cup M$. Put $K = S$, if $\mathrm{card}(S) = 2$, and $K = S \cup M$, if $\mathrm{card}(S) \leq 1$. We have proved $K \supseteq E(A)$.

For $a \in \overline{C^*}$ consider a set $B = \{h \in H;\ a = \sup_{\overline{\gamma}}\{x \in \overline{C};\ (x,h) \in \overline{\gamma}\}\}$. B belongs to $\mathcal{E}$ by Lemma 11, and as $B \cap K = \emptyset$, we see that $\deg(a) \leq 1$ for all $a \in \overline{C^*}$.

Suppose now that there exist $a, b \in \overline{C}$ such that $a \neq b$, $(a,b) \in \overline{\gamma}$ and $\deg(x) = 0$ for every $x \in \overline{C}$ with $(a,x) \in \overline{\gamma}$, $(x,b) \in \overline{\gamma}$ and $x \neq b$. Put $D = \{x \in \overline{C};\ (a,x) \in \overline{\gamma}$ and $(x,b) \in \overline{\gamma}\}$. Note that any $x \in D$, $x \neq b$, is in C. For every $h \in H$ there can be found $c \in \overline{C}$ such that $c = \sup_{\overline{\gamma}}\{y \in \overline{C};\ (y,h) \in \overline{\gamma}\}$. Thus by Lemma 11 $D \cap C$ belongs to $\mathcal{E}$ and for every $c, d \in D$ the set $\{x \in D;\ (d,x) \in \overline{\gamma},\ (x,c) \in \overline{\gamma}$ and $x \neq c\}$ also belongs to $\mathcal{E}$. If $b \notin C$, then $D \cap C$ has infinitely many elements and $E(A) = \emptyset$. Therefore $D \subseteq C$ can be assumed, and we see that $E(A) = \emptyset$ if $M \neq \{b\}$. Thus either there exist no $a, b \in \overline{C}$ with $a \neq b$, $(a,b) \in \overline{\gamma}$ and $\deg(x) = 0$ for any $x \in \overline{C}$ such that $(a,x) \in \overline{\gamma}$, $(x,b) \in \overline{\gamma}$ and $x \neq b$, or $M = \{b\}$ and $m = a$ is such that $(c,m) \in \gamma$ for all $c \in \overline{C^*}$ and $\deg(m) = 0$. Put $F = K$ in the former case, and $F = M \cap K$ in the latter case. We have proved that $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C}$ or $\overline{C^*}$, respectively. We have also proved that F contains $E(A)$, if some of the cases (i)–(iii) applies.

It remains to show $F = E(A)$. Take $k \in K$ and assume $k \notin J$ for some $J \in \mathcal{E}_2$. As $S = \emptyset$ implies $M = \emptyset$, and thus $F = \emptyset$, assume also $S \neq \emptyset$. Let $j, s \in J$ be such that $j \neq s$ and $s \in S$. As $j \in S$ provides $K \subseteq S$, we have $j \notin S$. For $j \in C$ we obtain $k \in J$ by $(j,k) \in \gamma$, $(s,k) \notin \gamma$ and by Lemma 11. Hence $J \cap C = \emptyset$. If $j \in H \backslash S$, then there can be found $c \in C$ with $(c,j) \notin \gamma$. As $(c,s) \in \gamma$, $c \in J$, again by Lemma 11. We have proved $S \cap J = \emptyset$.

Suppose now that $h, j \in J$ are such that $j \neq h$ and $h \in H \backslash S$. If $j \in C$, $s \in S$, then $(j, s) \in \gamma$, $(h, s) \notin \gamma$, and Lemma 11 provides $s \in J$. If $j \in H$, then the sets $\{a \in C;\ a \leq j\}$ and $\{a \in C;\ a \leq h\}$ are different by our degree assumption. Therefore we can assume that there exists $c \in C$ with $(c, j) \in \gamma$ and $(c, h) \notin \gamma$. From Lemma 11 we obtain $J \subseteq C$.

If $J \subseteq C$, and $a, b \in J$ are such that $(a, b) \in \gamma$ and $a \neq b$, note first that for any $c \in C$ with $(a, c) \in \gamma$, $(c, b) \in \gamma$ and $c \neq b$ we have $c \in J$ by $(b, c) \notin \gamma$ and Lemma 11. Consider now $x \in \overline{C}$ such that $(a, x) \in \gamma$, $(x, b) \in \gamma$ and $x \neq b$. If $\deg(x) = 1$, then there exists $h \in H$ with $(x, h) \in \overline{\gamma}$ and $(b, h) \notin \overline{\gamma}$. Thus $(a, h) \in \gamma$, $(b, h) \notin \gamma$, and hence from Lemma 11 we obtain $h \in J$, a contradiction with $J \subseteq C$. Therefore $\deg(x) = 0$ for any $x \in \overline{C}$ with $(a, x) \in \gamma$, $(x, b) \in \gamma$, $x \neq b$, and by the density assumption, $J = \{a, b\} = D$. □

Proposition 6. *Let $A = A(*)$ be a quasitrivial LD-groupoid with a non-trivial kernel η. $A(*)$ is subdirectly irreducible if and only if the following conditions are satisfied:*

(i) *There exists only one equivalence class of η with more than one element (denote this class by B).*
(ii) $\operatorname{card}(B) = 2$.
(iii) *The natural homomorphism $A \to A/\eta$ maps B to $E(A/\eta)$.*

If A is subdirectly irreducible, then $E(A) = B$.

Proof: Assume $E(A) \neq \emptyset$. By Lemma 16 η contains no class with three elements and at most one class with two elements. Hence there exists an equivalence class B as required by (i) and (ii). Identify A/η with $A' = (A \backslash B) \cup \{B\}$. If $C \in \mathcal{E}_2'$ and $B \notin C$, then $C \in \mathcal{E}_2$ and $E(A) = \emptyset$ by $B \in \mathcal{E}_2$. Therefore B has to be mapped inside $E(A')$.

On the other hand, let A be an LD-groupoid satisfying (i)–(iii). Then $B \subseteq E(A)$. If $C \in \mathcal{E}_2$ and $B \cap C = \emptyset$, then $C \in \mathcal{E}_2'$, a contradiction to $B \in E(A')$. Hence $B \cap C \neq \emptyset$ for every $C \in \mathcal{E}_2$. Assume now that $B = \{a, b\}$ and there exists $C \in \mathcal{E}_2$ with $a \in C$ and $b \notin C$. If $c \in C$ and $c \neq a$, then $(a, b) \in \gamma$ implies $(c, b) \in \gamma$ by Lemma 11. Similarly, $(b, c) \in \gamma$, and thus $(b, c) \in \eta$ and $b = c$. Therefore $B = E(A)$. □

From Proposition 5, Proposition 6 and Lemma 16 we obtain:

Corollary 2. *If $A = A(*)$ is a quasitrivial LD-groupoid, then* $\operatorname{card}(E(A)) \leq 2$.

Corollary 3. *A quasitrivial LD-groupoid $A(*)$ is simple iff* $\operatorname{card}(A) \leq 2$.

Proof: Every simple groupoid is subdirectly irreducible. If $A(*)$ is subdirectly irreducible and $\operatorname{card}(A) > 2$, then it contains a non-trivial congruence $\varepsilon_{E(A)}$. □

Propositions 5 and 6 together with Lemma 15 and Lemma 13 provide a complete characterization of subdirectly irreducible quasitrivial LD-groupoids.

By Proposition 5 there are subdirectly irreducible quasitrivial LD-groupoids for every cardinality κ. This contrasts with the case of both sided distributivity, in which every subdirectly irreducible quasitrivial groupoid contains at most four elements (observe that a quasitrivial LD-groupoid $A = A(*)$ is right distributive if and only if the set $B = \{b \in A$; there exists $a \in A$ with $(b, a) \in \gamma$ and $(b, a) \notin \eta\}$ is linearly ordered by γ, if $(b, a) \in \gamma$ for every $b \in B$ and $a \in A \backslash B$, and if $A \backslash B$ is either discrete, or a block of η).

By Proposition 3, for every subdirectly irreducible quasitrivial LD-groupoid $A = A(*)$ there exists a binary operation $\circ$ on A such that $A(*, \circ)$ is an LD-algebra.

The following problems seem to be open.

1. Is the variety generated by quasitrivial LD-groupoids characterized by the identities $a*(b*c) = (a*b)*(a*c)$, $a*a = a$, $(a*b)*b = a*b$ and $a*(a*b) = a*b$?
2. Which of the quasitrivial LD-groupoids are included in the variety of LD-groupoids generated by conjugation in groups (cf. [DKM])?
3. For which LD-groupoids $A(*)$ there can be defined a commutative associative operation $\circ$ on A such that $A(*, \circ)$ is an LD-algebra?

References

[DehAd] Dehornoy P., *The adjoint representations of left distributive structures*, Comm. Algebra **20** (1992), 1201–1215.

[DehBr] ______, *Braid groups and left distributive structures*, Transactions AMS, to appear.

[DKM] Drápal A., Kepka T., Musílek M., *Group conjugation has non-trivial LD-identities*, Comment. Math. Univ. Carolinae **35** (1994), 219–222.

[JeKe] Ježek J., Kepka T., *Quasitrivial and nearly quasitrivial distributive groupoids and semigroups*, Acta Univ. Carol. **19** (1978), 225–237.

[KepQ] Kepka T., *Quasitrivial groupoids and balanced identities*, Acta Univ. Carol. **22**, 49–64.

[LavFr] R. Laver, *The left distributive law and the freeness of an algebra of elementary embeddings*, Advances in Mathematics **91** (1992), 209-231.

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