A σ-porous set need not be σ-bilaterally porous

J. Nájares R., L. Zajíček*

Abstract. A closed subset of the real line which is right porous but is not σ-left-porous is constructed.

Keywords: sigma-porous, sigma-bilaterally-porous, right porous

Classification: Primary 26A99; Secondary 28A5

1. Introduction

Let $E \subset \mathbb{R}$ be a set, and let I be an interval. Then we denote by $\lambda(E, I)$ the length of the largest open subinterval of I which does not intersect E. The right porosity of E at $x \in \mathbb{R}$ is defined as

$$p^+(E, x) = \overline{\lim_{h \to 0_+}} \frac{\lambda\Big(E, (x, x+h)\Big)}{h}.$$

The left porosity $p^-(E, x)$ is defined by the symmetrical way.

We say that:

(i) E is right porous at x if $p^+(E, x) > 0$,
(ii) E is left porous at x if $p^-(E, x) > 0$,
(iii) E is bilaterally porous at x if it is porous both on the right and on the left at x.

The set E is said to be right (left, bilaterally) porous if it is right (left, bilaterally) porous at each of its points and σ-right-porous (σ-left-porous, σ-bilaterally-porous) if it is a countable union of right (left, bilaterally) porous sets. It is easy to see that a set is σ-bilaterally porous iff it is bilaterally σ-porous (i.e. it is both σ-right-porous and σ-left-porous). The main aim of the present article is to prove the following result.

Theorem. *There exists a closed set $F \subset \mathbb{R}$ which is right porous but is not σ-left-porous.*

We obtain the example slightly modifying the ideas of [F] and [Za 1].

We essentially use Lemma 5 which is a special case of the generalized Foran lemma [Za 3], which enables us to give a simple proof that our set F is not

*Supported by Research Grant GAUK 363.

σ-left-porous. Another ingredient of our proof is Proposition, which is analogous to Proposition 4.4 from [Za 2]. We believe that it can be also of some independent importance. Note that for symmetrical porosity an analogical proposition does not hold [E-H-S].

2. Proposition and lemmas

Definition 1. If $c > 0$, $M \subset \mathbb{R}$ and $r > 0$ are given, then we define

$$S(c, r, M) = \bigcup\{x \ominus (y-\sigma, y);\ y \in \mathbb{R},\ 0 < \sigma < r,\ (y-\sigma,\ y) \cap M \ = \emptyset\},$$

where $c \ominus (y-\sigma, y) = (y - c\sigma, y)$.

We shall need the following lemmas which are obvious.

Lemma 1. *If* $p^+(M, x) \geq c > 0$, *then* $x \in \bigcap\{S(\frac{2}{c}, r, M);\ r > 0\}$.

Lemma 2. *If* $c > 1$, $x \in M$ *and* $x \in \bigcap\{S(c, r, M);\ r > 0\}$, *then* $p^+(M, x) \geq \frac{1}{c}$.

Proposition. *Let A be a σ-right-porous set (σ-left-porous) and $c < 1$. Then there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *such that* $A = \bigcup\limits_{n=1}^{\infty} A_n$ *and* $p^+(A_n, x) \geq c$ *(*$p^-(A_n, x) \geq c$*, respectively) for any* $n \in \mathbb{N}$ *and* $x \in A_n$.

Proof: It is sufficient to give the proof for right porosity only. By definition $A = \bigcup\limits_{n=1}^{\infty} B_n$ where B_n is a right porous set for any $n \in \mathbb{N}$. Putting $B_{n,k} = \{x \in B_n;\ p^+(B_n, x) \geq \frac{1}{k}\}$ we have that $A = \bigcup\limits_{n,k=1,}^{\infty} B_{n,k}$ and $p^+(B_{n,k}, x) \geq \frac{1}{k}$ for any $x \in B_{n,k}$.

Thus it is sufficient to prove the following statement:

If $M \subset \mathbb{R}$, $a > 0$ *and, for each* $x \in M$, *the inequality* $p^+(M, x) \geq a$ *holds, then* $M = \bigcup\limits_{i=1}^{\infty} M_i$, *where* $p^+(M_i, y) \geq c$ *for any* $y \in M_i$.

We can suppose $a < c < 1$, the case $a \geq c$ being trivial. Choose $n \in \mathbb{N}$ such that $(\frac{1}{c})^n \geq \frac{2}{a}$ and define $C_k = M \cap \bigcap\limits_{r>0} S(c^{-k}, r, M)$. By Lemma 1 $M = C_n$ and therefore $M = \bigcup\limits_{k=2}^{n} \big(C_k \setminus C_{k-1}\big) \cup C_1$. By Lemma 2, we have $p^+(C_1, x) \geq c$ for any $x \in C_1$.

For $k = 2, \ldots, n$ define $T_{k,m} = C_k \setminus S(c^{-k+1}, m^{-1}, M)$. Then

$$\bigcup_{m=1}^{\infty} T_{k,m} = C_k \setminus \bigcap_{m=1}^{\infty} S(c^{-k+1}, m^{-1}, M) = C_k \setminus C_{k-1}.$$

Since $T_{k,m} \subset C_k$, for each $z \in T_{k,m}$ and $r > 0$, there exist y and t such that $0 < t < \min(r, m^{-1})$, $(y-t, y) \cap M = \emptyset$ and $z \in c^{-k} \ominus (y-t, y)$. Put $J = c^{-k+1} \ominus (y-t, y)$. Then $z \in c^{-1} \ominus J$ and $J \cap T_{k,m} = \emptyset$, since $J \subset S(c^{-k+1}, m^{-1}, M)$.

Thus, for each $z \in T_{k,m}$, we have $z \in \bigcap_{r>0} S(c^{-1}, c^{-k+1}r, T_{k,m})$ and therefore $p^+(T_{k,m}, z) \geq c$ by Lemma 2, which proves our statement. $\square$

For the sake of brevity, in the following we shall say that E is V-porous at x if $p^-(E,x) > \frac{100}{101}$. The following lemma is easy to prove.

Lemma 3. *Let $E \subset \mathbb{R}$, $x \in \mathbb{R}$ and a natural number p be given such that $x - 10^{-k}$ or $x - 10^{-(k+1)}$ belongs to E for each natural $k > p$. Then E is not V-porous at x.*

The following lemma is an immediate consequence of Proposition.

Lemma 4. *A set $E \subset \mathbb{R}$ is σ-left-porous iff it is σ-V-porous.*

Definition 2. We say that $\mathcal{F} \subset exp\, \mathbb{R}$ is a non-σ-V-porosity family if the following conditions hold:

(a) $\mathcal{F}$ is a nonempty family of nonempty closed sets,
(b) for each $F \in \mathcal{F}$ and each open set $G \subset \mathbb{R}$ with $F \cap G \neq \emptyset$, there exists $F^* \in \mathcal{F}$ such that $\emptyset \neq F^* \cap G \subset F \cap G$ and F is V-porous at no point of $F^* \cap G$.

We shall need the following lemma which is a special case of [Za 3, Lemma 4.3].

Lemma 5. *Let $\mathcal{F}$ be a non-σ-V-porosity family. Then no set from $\mathcal{F}$ is σ-V-porous.*

3. Proof of theorem

Our theorem stated in Introduction immediately follows from Lemma 7 and Lemma 8 below. To formulate them, we need some notions.

Definition 3. Let $x \in (0,1)$. As usual, we write $x = 0, a_1 a_2 ...$ if $x = \sum_{i=1}^{\infty} a_i 10^{-i}$ and $a_i \in \{0, 1, ..., 9\}$. The uniqueness of the expansion is obtained using terminating 0's whenever x has two expansions. Let $a \in \{0, 1, ..., 9\}$ be a digit. The density and the upper density of a in the expansion of x are defined as

$$d(a,x) = \lim_{n\to\infty} \frac{\#\{k; 1 \leq k \leq n, \quad a_k(x) = a\}}{n},$$
$$\bar{d}(a,x) = \overline{\lim_{n\to\infty}} \frac{\#\{k; 1 \leq k \leq n, \quad a_k(x) = a\}}{n}.$$

The following easy fact is well known and easy to prove.

Lemma 6. *The function $x \mapsto \bar{d}(a,x)$ is Borel measurable on $(0,1)$.*

Definition 4. For a natural n and $x \in (0,1)$ put

$$c(x,n) = \#\{k;\ n^2 < k \leq (n+1)^2, a_k(x) = 9\} \quad \text{and}$$
$$e(x,n) = \#\{k;\ n^2 < k \leq (n+1)^2, a_k(x) \neq 9\}$$

Let a natural number N, $\varepsilon > 0$, $1 > \alpha > 0$ and digits $a_1, \dots, a_{N^2} \in \{0, 1, ..., 9\}$ be given. Then we define the set $A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$ as the set of all $x \in (0, 1)$ for which

$$a_1(x) = a_1, ..., a_{N^2}(x) = a_{N^2} \quad \text{and} \tag{1}$$

$$1 - \frac{\varepsilon}{n^\alpha} \leq \frac{c(x, n)}{2n + 1} < 1 \quad \text{whenever} \quad n \geq N \tag{2}$$

Lemma 7. *Let* $0 < \alpha < 1$, $\varepsilon > 0$ *and digits* $a_1, \dots, a_{N^2} \in \{0, 1, ..., 9\}$ *such that*

$$N > \max[(1 + \varepsilon)^{\frac{1}{\alpha}}, \varepsilon^{\frac{1}{\alpha - 1}}] \tag{3}$$

be given.

Then $A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$ *is a closed set which is not* σ*-left-porous.*

Proof: Obviously (2) implies that

$$e(x, n) \neq 0 \quad \text{whenever} \quad n \geq N \quad \text{and} \quad x \in A(\alpha, a_1, ..., a_{N^2}, \varepsilon). \tag{4}$$

Now suppose that $x_n \in A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$ and $x_n \to x$. On account of (4) we easily obtain that

$$(a_1(x_n), a_2(x_n), \dots) \to (a_1(x), a_2(x), \dots)$$

in the space $\mathbb{N}^\mathbb{N}$ and consequently $x \in A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$. Thus we have that $A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$ is closed.

The condition (2) is equivalent to

$$c(x, n) \in \Big[(1 - \frac{\varepsilon}{n^\alpha})(2n + 1),\ 2n + 1\Big) := I_n \quad \text{for} \quad n \geq N.$$

If $n \geq N$, we have by (3)

$$(1 - \frac{\varepsilon}{n^\alpha})(2n + 1) > (1 - \frac{\varepsilon}{1 + \varepsilon})(2(1 + \varepsilon)^{\frac{1}{\alpha}}) > 2 \qquad \text{and}$$
$$(2n + 1) - (1 - \frac{\varepsilon}{n^\alpha})(2n + 1) = \frac{\varepsilon}{n^\alpha}(2n + 1) > 2\varepsilon n^{1-\alpha} > 2.$$

Thus we have $I_n \subset (2, 2n + 1)$ and *length* $(I_n) > 2$ for $n \geq N$; consequently $A(\alpha, a_1, ..., a_{N^2}, \varepsilon) \neq \emptyset$ and $c(x, n) > 2$ whenever $x \in A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$ and $n \geq N$.

Now let $\mathcal{F}$ denote the family of all sets of the form $A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$ for which (3) holds. By Lemma 4 and Lemma 5 it is sufficient to prove that $\mathcal{F}$ is a non-σ-V-porosity family. To this end suppose that $F = A(\alpha, a_1, ..., a_{N^2}, \varepsilon) \in \mathcal{F}$ and an open set $G \subset \mathbb{R}$ such that $F \cap G \neq \emptyset$ are given.

Choose an arbitrary $y \in F \cap G$ and find a natural M so large that

(5) $$M > N, \quad M > (\frac{\varepsilon}{2})^{\frac{1}{\alpha-1}} \quad \text{and}$$

$$F^* := A(\alpha, a_1, ..., a_{N^2}, a_{N^2+1}(y), ..., a_{M^2}(y), \frac{1}{2}\varepsilon) \subset G.$$

Clearly $F^* \subset F$. On account of (3) and (5) we have

$$M > \max\left((1+\frac{\varepsilon}{2})^{\frac{1}{\alpha}}, (\frac{\varepsilon}{2})^{\frac{1}{\alpha-1}}\right)$$

and therefore $F^* \in \mathcal{F}$. Thus it is sufficient to prove that F is V-porous at no point $z \in F^*$. To prove this, fix an arbitrary $z \in F^*$ and consider an arbitrary natural $k > (M+1)^2$. By Lemma 3 it is sufficient to prove that at least one of the points $z_k^- = z - 10^{-k}$, $z_{k+1}^- = z - 10^{-(k+1)}$ belongs to F. It is easy to see that

(6) $$c(z,n) - 1 \le c(z_k^-, n) \quad \text{and} \quad c(z,n) - 1 \le c(z_{k+1}^-, n), \quad \text{for each } n.$$

Since $z \in F^*$, we have $c(z,M) > 0$ (we know even $c(z,M) > 2$) and therefore

(7) $$a_s(z) = a_s(z_k^-) = a_s(z_{k+1}^-) \quad \text{for} \quad s \le M^2.$$

Now suppose that $x \in \{z_k^-, z_{k+1}^-\}$. Then (7) says that

$$a_s(x) = a_s(z) \quad \text{for} \quad s \le M^2.$$

For $n \ge M$ the definition of F^*, (6) and (5) yield

$$\frac{c(x,n)}{2n+1} \ge \frac{c(z,n)-1}{2n+1} \ge 1 - \frac{\varepsilon}{2n^\alpha} - \frac{1}{2n+1} > 1 - \frac{\varepsilon}{n^\alpha}.$$

Thus it is sufficient to establish that, for $x = z_k^-$ or $x = z_{k+1}^-$,

(8) $$e(x,n) \ne 0, \quad \text{for each} \quad n \ge M.$$

To this end suppose that

$$e(z_k^-, n) = 0 \quad \text{for some} \quad n \ge M.$$

Since $c(z,n) \ne 0$, this condition easily implies that

$$k = n^2 + i \quad \text{where} \quad i \in \{1, ..., 2n\},$$
$$a_{n^2+1}(z) = 0, ..., a_{n^2+i}(z) = 0 \quad \text{and}$$
$$a_{n^2+i+1}(z) = 9, ..., a_{(n+1)^2}(z) = 9.$$

Consequently it is easy to see that (8) holds for $x = z_{k+1}^-$. □

Lemma 8. *If* $\frac{1}{2} < \alpha < 1$, *then the set* $A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$ *from Lemma 7 is right porous.*

Proof: Choose an arbitrary $x \in A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$.

For each natural n, let m_n be the maximum of those natural i, for which there exist natural numbers u, v such that

$$n^2 \leq u < v \leq (n+1)^2 \ , \ a_s(x) = 9 \ \text{ for each } \ u < s \leq v \tag{9}$$

and $v - u = i$. It is easy to see that

$$2n + 1 - e(x,n) = c(x,n) \leq m_n(e(x,n) + 1) \ \text{ and consequently}$$

$$m_n \geq \frac{2n + 1 - e(x,n)}{e(x,n) + 1}. \tag{10}$$

On account of (2) we have that

$$e(x,n) \leq \frac{\varepsilon(2n+1)}{n^\alpha} \ \text{ for } \ n \geq N$$

and therefore (10) implies that there exists $c > 0$ and a natural n_0 such that

$$m_n \geq cn^\alpha \ \text{ for all } \ n \geq n_0. \tag{11}$$

Now, for each n, choose u_n, v_n such that

$$v_n - u_n = m_n \ \text{ and (9) holds for } \ u = u_n, \ v = v_n.$$

Put

$$y_n = x + 10^{-v_n} \quad \text{and} \quad z_n = x + 10^{-v_n + 1}.$$

It is easy to see that, for each $t \in (y_n, z_n)$, we have

$$a_s(t) = 0, \quad \text{for each } \ u_n < s \leq v_n - 1$$

and therefore

$$c(t,n) \leq 2n + 1 - (m_n - 1). \tag{12}$$

If n is so big that $n > n_0$, $n > N$ and $2n + 2 - cn^\alpha < (2n+1)(1 - \frac{\varepsilon}{n^\alpha})$, we have by (12) and (11)

$$c(t,n) \leq 2n + 2 - cn^\alpha < (2n+1)(1 - \frac{\varepsilon}{n^\alpha}).$$

Thus we obtain by (2) that $t \notin A(\alpha, a_1, ..., a_{N^2}, \varepsilon)$. Consequently

$$p^+\Big(A(\alpha, a_1, ..., a_{N^2}, \varepsilon), x\Big) \geq \overline{\lim_{n \to \infty}} \frac{10^{-(v_n - 1)} - 10^{-v_n}}{10^{-v_n + 1}} = \frac{9}{10}.$$

References

[F] Foran J., *Continuous functions need not have σ-porous graphs*, Real Anal. Exchange **11** (1985–86), 194–203.

[Za 1] Zajíček L., *On σ-porous sets and Borel sets*, Topology Appl. **33** (1989), 99–103.

[Za 2] ———, *Sets of σ-porosity and sets of σ-porosity (q)*, Časopis Pěst. Mat. **101** (1976), 350–359.

[Za 3] ———, *Porosity and σ-porosity*, Real Anal. Exchange **13** (1987–88), 314–350.

[E-H-S] Evans M.J., Humke P.D., Saxe K., *A symmetric porosity conjecture of L. Zajíček*, Real Anal. Exchange **17** (1991–92), 258–271.

Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

(*Received* January 19, 1994)