A primrose path from Krull to Zorn

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Abstract. Given a set X of "indeterminates" and a field F, an ideal in the polynomial ring $R = F[X]$ is called conservative if it contains with any polynomial all of its monomials. The map $S \mapsto RS$ yields an isomorphism between the power set $\mathscr{P}(X)$ and the complete lattice of all conservative prime ideals of R. Moreover, the members of any system $\mathscr{S} \subseteq \mathscr{P}(X)$ of finite character are in one-to-one correspondence with the conservative prime ideals contained in $P_{\mathscr{S}} = \bigcup\{RS : S \in \mathscr{S}\}$, and the maximal members of $\mathscr{S}$ correspond to the maximal ideals contained in $P_{\mathscr{S}}$. This establishes, in a straightforward way, a "local version" of the known fact that the Axiom of Choice is equivalent to the existence of maximal ideals in non-trivial (unique factorization) rings.

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In 1979, Hodges [3] derived a certain maximal principle on trees, equivalent to Zorn's Lemma and hence to the Axiom of Choice (AC), from the statement that every nontrivial unique factorization domain contains a maximal ideal. In fact, he showed more, namely that if suffices to take into account certain "pseudo-localizations" of polynomial rings (in an arbitrary number of indeterminates) over the rational number field $\mathscr{Q}$. Recently, Banaschewski [1] gave a short and direct deduction of AC from the above specific maximal ideal theorem. Since one argument in his proof involved the infinity of $\mathscr{Q}$, he asked whether an alternative argument might provide the same conclusion over an arbitrary (possibly finite) base field F. We shall show that this is in fact the case, by establishing an elementary one-to-one correspondence between the subsets of a fixed set X and so-called *conservative* prime ideals of the polynomial ring $R = F[X]$. Concerning basic ring-theoretical background, see, for example, the monograph "*Commutative rings*" by Kaplansky [4].

By a *prime set* in an arbitrary ring, we mean a proper subset P such that $ab \in P$ if and only if $a \in P$ or $b \in P$. Hence one version of Krull's Prime Ideal Theorem states that every ideal contained in a prime set P is contained in a prime ideal $Q \subseteq P$. The equivalence of this statement, even for non-commutative rings, with the lattice-theoretical Prime Ideal Theorem (PIT), alias Boolean Ultrafilter Theorem, has been established in [2]. (Notice, however, that in the non-commutative case, a prime ideal need not be a prime set.) By the work of Halpern and Levy, PIT is weaker than AC in BNG set theory (cf. [6, p. 99]).

Henceforth, we focus on the following specific setting: given a set X and an arbitrary field F, we are considering the (commutative) polynomial ring $R = F[X]$ with the elements of X as indeterminates. The multiplicative submonoid generated by these indeterminates is the free abelian monoid over X. It consists of all (*unitary*) *monomials* and will be denoted by M. Recall that any polynomial $a \in R$ has a unique representation $q_1 m_1 + \cdots + q_n m_n$ as a linear combination of monomials $m_1, \ldots, m_n$ with non-zero coefficients $q_1, \ldots, q_n \in F$. The collection of these *a-monomials* is denoted by M_a. For any subset A of R, we put $M_A = \bigcup\{M_a; a \in A\}$ and call A (M-)*conservative* if $M_A \subseteq A$. Writing RS for the ideal generated by a subset S of R, one immediately observes that an ideal I is conservative iff it is of the form RS for some $S \subseteq M$ (in fact, $I = RM_I$).

The conservative ideals of R form a closure system $\mathscr{CI}(R)$, hence a complete lattice. The corresponding closure operator assigns to each $A \subseteq R$ the ideal RM_A. The lattice $\mathscr{CI}(R)$ is easily seen to be superalgebraic, that is, algebraic and completely distributive: indeed , each conservative ideal I is a join of completely join-prime (= supercompact) members of $\mathscr{CI}(R)$, namely of the principal ideals generated by monomials in I. Furthermore, not only the join of conservative ideals is conservative, but also the product of any two conservative ideals. In other words, $\mathscr{CI}(R)$ is a subquantale of the quantale $\mathscr{I}(R)$ of all ideals (see, for example, [5]). Moreover, the map $S \mapsto RS$ yields an isomorphism between the Alexandrov topology of all ideals of the monoid M (i.e. of all subsets S of M with $mS \subseteq S$ for all $m \in M$) and $\mathscr{CI}(R)$. The inverse isomorphism is given by $I \mapsto M_I = M \cap I$. Next, we characterize the ideals of the form RS where S is a set of indeterminates.

Lemma 1. *The assignment $S \mapsto RS$ yields an isomorphism between the power set $\mathscr{P}(X)$ and the complete lattice of all conservative prime ideals.*

Proof: It is easily verified that each set RS with $S \subseteq X$ is a conservative prime ideal. Conversely, let P be any conservative prime ideal of R. Then, for $a \in P$, each a-monomial m belongs to P, and as P is prime, $m = rx$ for some $r \in R$ and $x \in S = X \cap P$. Hence the element a is a member of the ideal RS, being a linear combination of its monomials. This proves the inclusion $P \subseteq RS$, and the converse inclusion is clear since P is an ideal containing S. The equation

$$S = X \cap RS \qquad (S \subseteq X)$$

shows that the map $S \mapsto RS$ is one-to-one, with inverse $P \mapsto X \cap P$. Of course, these two mutually inverse maps preserve inclusion and are therefore isomorphisms.

□

By a *primrose* of R, we mean a subset P of R such that for each $a \in P$, there is some $S \subseteq X$ with $a \in RS \subseteq P$. In view of Lemma 1, the primroses are just the unions of conservative prime ideals, in other words, sets of the form

$$P_{\mathscr{S}} = \bigcup\{RS : S \in \mathscr{S}\}$$

with $\mathscr{S} \subseteq \mathscr{P}(X)$. Clearly, any such union is still a conservative prime set, but the converse does not hold. For example, if x and y are distinct indeterminates from X then the union $P = Rx \cup Ry \cup R(x+y)$ is a conservative prime set but not a primrose since there is no $S \subseteq X$ such that $x + y \in RS \subseteq P$.

Recall that a collection $\mathscr{S}$ of subsets of X is a *system of finite character* (on X) provided a set S belongs to $\mathscr{S}$ if and only if $E \in \mathscr{S}$ for all finite subsets E of S. Among the various maximal principles equivalent to the Axiom of Choice (cf. [6]), the most convenient version is here the lemma of Tukey-Teichmüller, stating that any member of a system of finite character is contained in a maximal one.

Lemma 2. *There is a one-to-one correspondence $\mathscr{S} \mapsto P_{\mathscr{S}}$ between the systems of finite character on X and the primroses of R. Moreover, for fixed $\mathscr{S}$, the map $S \mapsto RS$ induces a bijection between $\mathscr{S}$ and the set of all conservative prime ideals contained in $P_{\mathscr{S}}$.*

Proof: Given any primrose P, it is straightforward to show that the system

$$\mathscr{S}_P = \{S \subseteq X : RS \subseteq P\}$$

is of finite character, and $P = P_{\mathscr{S}_P}$.

Clearly, if $\mathscr{S} \subseteq \mathscr{P}(X)$ is any system of finite character with $P = P_{\mathscr{S}}$ then we have $\mathscr{S} \subseteq \mathscr{S}_P$. On the other hand, if S is a member of $\mathscr{S}_P$ then for each finite subset $E = \{x_1, \ldots, x_n\}$ of S, the element $x_1 + \cdots + x_n$ belongs to $RS \subseteq P = P_{\mathscr{S}}$, hence to RS' for some $S' \in \mathscr{S}$, so that by Lemma 1, $E \subseteq S'$. Thus $E \in \mathscr{S}$ for each finite $E \subseteq S$, and so $S \in \mathscr{S}$. This proves the equation $\mathscr{S} = \mathscr{S}_P$ and shows that the map $P \mapsto \mathscr{S}_P$ is inverse to the map $\mathscr{S} \mapsto P_{\mathscr{S}}$. □

We now come to a key result.

Lemma 3. *For any primrose P and any ideal $I \subseteq P$, the smallest conservative ideal containing I is still a subset of P.*

Proof: First, we prove the inclusion $Rm + I \subseteq P$ for $a \in I$ and any a-monomial m. Let $b \in I$ and choose an exponent n large enough such that no b-monomial has m^n as a factor. Then $c = m^n a + b \in I \subseteq P$, hence $c \in RS \subseteq P$ for some $S \subseteq X$. As m^{n+1} and all b-monomials are c-monomials, too, one obtains $m^{n+1} \in RS$ and $M_b \subseteq RS$. But RS is a prime ideal by Lemma 1, so that $Rm + b \subseteq RS \subseteq P$.

Now it is easy to show that the conservative ideal RM_I is a subset of P: for any finite subset E of M_I, a straightforward induction gives $RE + I \subseteq P$, and then it follows that $RM_I \subseteq P$. □

Corollary. *Any ideal maximal among the ideals contained in a fixed primrose P is a conservative prime ideal.*

For any prime set $P \subseteq R$, the quotients $\frac{r}{u}$ with $r \in R$ and $u \in R \setminus P$ form a subring R_P of the quotient field of R, and the canonical embedding of R in R_P gives rise to a one-to-one correspondence between the prime ideals of R contained in P and the prime ideals of R_P (cf. [5, 1–5]). We shall refer to R_P as a *pseudo-localization* of R. In all, we have established the following

Proposition. *Let X be a set, F an arbitrary field, and R the polynomial ring $F[X]$. Then the maximal members of any system $\mathscr{S}$ of finite character on X are in one-to-one correspondence with the maximal ideals contained in $P_{\mathscr{S}}$, and consequently, with the maximal ideals of the pseudo-localization $R_{P_{\mathscr{S}}}$.*

This immediately leads to a "local version" of Hodges' result that the existence of maximal ideals in unique factorization rings of the above type implies the Axiom of Choice.

Corollary. *The following two statements on a set X and a polynomial ring $R = F[X]$ are equivalent:*

(a) *Each system of finite character on X has a maximal member.*
(b) *Each pseudo-localization R_P by a primrose P has a maximal ideal.*

Notice that (a) entails the existence of a set of representatives for any partition $\mathscr{A}$ of X, since any such set is a maximal member of the following system of finite character:

$$\mathscr{S} = \{S \subseteq X : |S \cap A| \leq 1 \ \text{ for each } \ A \in \mathscr{A}\}.$$

Corollary. *Under the assumption of* PIT, *for any ideal I contained in a primrose P, there is a conservative prime ideal RS with $I \subseteq RS \subseteq P$.*

Proof: The set of all conservative ideals contained in P is closed under directed unions, and its complement is multiplicatively closed in $\mathscr{C}\mathscr{I}(R)$. Hence, by the Separation Lemma for Quantales which is equivalent to PIT (see [2]), any conservative ideal $I \subseteq P$ is contained in a conservative prime ideal $RS \subseteq P$, and Lemma 3 completes the proof. □

Added in proof. It can be shown that PIT is not only sufficient but also necessary for the above conclusion.

References

[1] Banaschewski B., *A new proof that "Krull implies Zorn"*, preprint, McMaster University, Hamilton, 1993.
[2] Banaschewski B., Erné M., *On Krull's separation lemma*, Order **10** (1993), 253–260.
[3] Hodges W., *Krull implies Zorn*, J. London Math. Soc. **19** (1979), 285–287.
[4] Kaplansky I., *Commutative Rings*, The University of Chicago Press, Chicago, 1974.
[5] Rosenthal K., *Quantales and Their Applications*, Pitman Research Notes in Mathematics Series **234**, Longman Scientific and Technical, Essex, 1990.
[6] Rubin H., Rubin J.E., *Equivalents of the Axiom of Choice, II*, North-Holland, Amsterdam-New York-Oxford, 1985.

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