A fixed point theorem in a Hausdorff topological vector space

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Abstract. In this paper, we will give a new fixed point theorem for lower semicontinuous multimaps in a Hausdorff topological vector space.

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1. Introduction

In 1912, Brouwer proved his well-known fixed point theorem and next Schauder extended the validity of Brouwer's fixed point theorem to normed linear space. This was further generalized by Tychonoff by showing that Schauder's proof could be adopted to prove the existence of fixed point even if the domain lies in a locally convex Hausdorff topological vector space instead of a normed linear space.

On the other hand, Kakutani generalized Brouwer's fixed point theorem to multimaps and applied the result to prove a version of the von Neumann minimax principle in R^n. Next Kakutani's theorem was also extended to Banach spaces by Bohnenblust-Karlin and to locally convex spaces by Ky Fan and Glicksberg.

Till now, there have been a number of attempts to generalize the Schauder-Tychonoff fixed point theorem or Fan-Glicksberg fixed point theorem in several directions and there also have been a number of applications in various fields of nonlinear analysis (e.g. [1], [4], [7]). It is an outstanding conjecture that the Schauder-Tychonoff fixed point theorem can be generalized in a Hausdorff topological vector space [2].

The purpose of this paper is to give a new fixed point theorem for lower semicontinuous multimaps in a Hausdorff topological vector space with additional topological condition, which can be useful in many applications.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by A^o, $cl\, A$ the interior of A and the closure of A in X, respectively. If A is a subset of a vector space E, we shall denote by $co\, A$ the

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convex hull of A. For $A, B \subset E$, and a real number t, the following notations will be used:

$$A + B := \{a + b \in E \mid a \in A, b \in B\}, \quad t \cdot A := \{ta \in E \mid a \in A\}.$$

Let X, Y be non-empty topological spaces and $T : X \to 2^Y$ be a multimap. The multimap T is said to be *open* or have *open graph* if the graph of T (Gr $T = \{(x, T(x)) \in X \times Y \mid x \in X\}$) is open in $X \times Y$. We may call $T(x)$ the *upper section* of T, and $T^{-1}(y) = (\{x \in X \mid y \in T(x)\})$ the *lower section* of T. It is easy to check that if T has open graph, then the upper and lower sections of T are open; however the converse is not true in general (see [8, p. 104]). A multimap $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$; and a multimap $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, then there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. A multimap T is said to be *continuous* if T is lower semicontinuous and upper semicontinuous. And it is noted that when T is single-valued, then the either form of semicontinuity is equivalent to continuity as a function.

We begin with the following lemma, which is useful to relax the assumption of having open graph property or having open lower sections.

Lemma 1. *Let X be a non-empty set in a Hausdorff topological space, E a Hausdorff topological vector space and Y be a non-empty subset of E. If a multimap $T : X \to 2^Y$ is lower semicontinuous and U is a non-empty open subset of E, then the multimap $T_U : X \to 2^Y$, defined by*

$$T_U(x) = (T(x) + U) \cap Y, \qquad \text{for each } x \in X,$$

has the open graph in $X \times Y$.

Proof: Let $(x_i, y_i)_{i \in I} \notin$ Gr T_U be a net in $X \times Y$, which converges to $(x, y) \in X \times Y$; i.e. $y_i \notin T(x_i) + U$ for all $i \in I$. Then it suffices to show that $(x, y) \notin$ Gr T_U.

Suppose the contrary, i.e. $(x, y) \in$ Gr T_U. Then there exist $s \in T(x)$, $u \in U$ such that $y = s + u \in Y$. Since U is open, we can find a balanced open neighborhood W of 0 in E such that $u + W + W \subset U$. Since T is lower semicontinuous and $s \in T(x)$, there exists an open neighborhood N_x of x such that $T(z) \cap (s + W) \neq \emptyset$ for all $z \in N_x$. Since $(x_i), (y_i)$ converge to x, y, respectively, there exists $i_o \in I$ such that $T(x_i) \cap (s + W) \neq \emptyset$ and $y_i \in y + W$ for all $i \geq i_o$.

For any fixed $j \geq i_o$, $s + w' \in T(x_j)$ for some $w' \in W$ and $y_j = y + w''$ for some $w'' \in W$. Then we have

$$\begin{aligned} T(x_j) &\ni s + w' \\ &= s + u - u + w'' - w'' + w' \\ &= y_j - u + w' - w''. \end{aligned}$$

Since W is a balanced neighborhood of 0, $y_j \in T(x_j) + u + W + W \subset T(x_j) + U$, which contradicts the assumption. This completes the proof. $\square$

Remark. It should be noted that the lower semicontinuity of T is essential in Lemma 1. In fact, for an upper semicontinuous multimap $T : X = [0,3] \to 2^X$ defined by

$$T(x) = \begin{cases} \{2\}, & \text{if } \ 0 \le x < 1, \\ [1,2], & \text{if } \ x = 1, \\ \{1\}, & \text{if } \ 1 < x \le 3, \end{cases}$$

and an open set $U := (-\frac{1}{3}, \frac{1}{3}) \subset R^1$, it is easy to see that the multimap T_U does not have open graph in $X \times X$.

The following Fan-Browder fixed point theorem [3, Theorem 10.3.6] is an essential tool for proving our main result.

Lemma 2. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space and a multimap $T : X \to 2^X$ satisfy the following:*

(i) *for each $x \in X$, $T(x)$ is non-empty convex,*
(ii) *for each $y \in X$, $T^{-1}(y)$ is open.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

3. Main results

We shall prove the following new fixed point theorem for lower semicontinuous multimaps in a Hausdorff topological vector space:

Theorem 1. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E and $T : X \to 2^X$ be a lower semicontinuous multimap such that each $T(x)$ is non-empty convex (not necessarily closed). Furthermore assume the following:*

If $y \notin (T(y) + U) \cap X$ for some open neighborhood U of 0 in E, then there exists an open neighborhood V of 0 in E such that

$$(*) \qquad y \notin cl\,\{x \in X \mid x \in (T(x) + co\,V) \cap X\}.$$

Then there exists a point $\tilde{x} \in X$ such that $\tilde{x} \in T(\tilde{x})$.

Proof: Let $\mathcal{B}$ be a local basis of open neighborhoods of 0 in E, and $U \in \mathcal{B}$ be arbitrarily given. Then $co\,U$ is an open convex neighborhood of 0. Since T is lower semicontinuous, by Lemma 1, the correspondence $T_U : X \to 2^X$, defined by

$$T_U(x) = (T(x) + co\,U) \cap X, \quad \text{for each } \ x \in X,$$

has an open graph in $X \times X$. Therefore $T_U^{-1}(y)$ is open for each $y \in X$.

Since each $T(x)$ is non-empty convex, $T_U(x)$ is non-empty convex for each $x \in X$. Therefore, by Lemma 2, there exists a point $\hat{x} \in X$ such that $\hat{x} \in T_U(\hat{x})$.

Now we shall use the following notations: For each $U \in \mathcal{B}$,

$$\mathcal{F}_U := \{x \in X \mid x \in (T(x) + co\, U) \cap X\},$$
$$\mathcal{F}'_U := \{x \in X \mid x \in (T(x) + U) \cap X\}.$$

Then $\mathcal{F}_U$ is non-empty and $\mathcal{F}'_U \subseteq \mathcal{F}_U$ for each $U \in \mathcal{B}$. It is clear that the family $\{\mathcal{F}_U \mid U \in \mathcal{B}\}$ of fixed point set of T_U has the finite intersection property. Therefore we have

$$\bigcap_{U\in\mathcal{B}} cl\, F_U \neq \emptyset.$$

We now claim that

$$\bigcap_{U\in\mathcal{B}} cl\, F_U = \bigcap_{U\in\mathcal{B}} F'_U.$$

In fact, it is clear that $\bigcap_{U\in\mathcal{B}} cl\, F_U \supseteq \bigcap_{U\in\mathcal{B}} F'_U$. For the reverse inclusion, if $y \notin \bigcap_{U\in\mathcal{B}} F'_U$, then $y \notin (T(y) + U) \cap X$ for some $U \in \mathcal{B}$. By the assumption $(*)$, there exists $V \in \mathcal{B}$ such that $y \notin cl\, \mathcal{F}_V$, i.e. $y \notin \bigcap_{V\in\mathcal{B}} cl\, F_V$. Therefore we have

$$\bigcap_{U\in\mathcal{B}} F'_U = \bigcap_{U\in\mathcal{B}} cl\, F_U \neq \emptyset;$$

so that there exists $\tilde{x} \in \bigcap_{U\in\mathcal{B}} F'_U$. Then $\tilde{x}$ is the desired fixed point of T; i.e.

$$\begin{aligned}\tilde{x} &\in \bigcap_{U\in\mathcal{B}} (T(\tilde{x}) + U) \cap X\\ &= (T(\tilde{x}) + \bigcap_{U\in\mathcal{B}} U) \cap X\\ &= T(\tilde{x}),\end{aligned}$$

which completes the proof. □

Remarks. (i) When E is a locally convex space and the map $T = f$ is single-valued, the assumption $(*)$ is automatically satisfied. In fact, since E is locally convex, we can choose a local basis $\mathcal{B}$ of 0 in E, whose elements are open convex neighborhoods of 0. Then for each $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V \subset cl\, V \subset U$. If $y \notin (f(y) + U) \cap X$ for some open neighborhood U of 0 in E, then $y \notin (f(y) + cl\, V) \cap X$. And it is easy to see that the fixed point set $\{x \in X \mid x \in (f(x) + cl\, V) \cap X\}$ is closed. Therefore we have that $y \notin cl\,\{x \in X \mid x \in (f(x) + co\, V) \cap X\}$; so that the assumption $(*)$ is satisfied.

In this case, we can obtain the Tychonoff fixed point theorem as a corollary.

(ii) Note that the assumption $(*)$ means the transfer-closed valued property of the map $x \to (T(x) + co\, U) \cap X$, and this assures the non-empty intersection property of the fixed point sets (e.g. see [7]).

The above Theorem 1 is a new fixed point theorem in a Hausdorff topological vector space for lower semicontinuous multimaps. It should be noted that when the multimap satisfies the assumption $(*)$, we can obtain a number of applications of Theorem 1 in Hausdorff topological vector spaces by replacing the Schauder-Tychonoff fixed point theorem by the above Theorem 1 as an essential tool for proving existence results.

The following example shows that Theorem 1 is a new fixed point theorem in Hausdorff topological vector space, which is comparable to Fan-Browder's fixed point theorem.

Example. Let $X = [0, 2]$ be a compact convex subset of R^1 and $T : X \to 2^X$ be a multimap defined by

$$T(x) = \begin{cases} (1, 2-x), & \text{if} \quad 0 \le x < 1, \\ \{1\}, & \text{if} \quad x = 1, \\ [1-\frac{x}{2}, 1), & \text{if} \quad 1 < x \le 2. \end{cases}$$

Then each $T(x)$ and $T^{-1}(y)$ are not necessarily closed nor open for each $x, y \in X$. Therefore neither Fan-Browder's fixed point theorem nor any selection theorem (e.g. see [1]) can be applicable to this setting. However, since we can easily check the assumption $(*)$ [e.g. for an open set $U = (-\frac{1}{n}, \frac{1}{n})$ with sufficiently large n, we can choose an appropriate neighborhood $V = (-\frac{1}{2n}, \frac{1}{2n})$] and T is lower semicontinuous, by Theorem 1 there exists a fixed point $1 \in X$ such that $1 \in T(1)$.

The Fan-Browder fixed point theorem, i.e. Lemma 2, cannot have a single-valued version; but when T is single-valued in Theorem 1, T is continuous as a function, so that Theorem 1 can be stated as follows:

Theorem 2. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E and $f : X \to X$ be a continuous function. Furthermore assume the following:*

If $y - f(y) \notin U$ for some open neighborhood U of 0 in E, then there exists an open neighborhood V of 0 in E such that

$$y \notin cl\,\{x \in X \mid x \in (f(x) + co\,V) \cap X\}.$$

Then there exists a point $\hat{x} \in X$ such that $\hat{x} = f(\hat{x})$.

Corollary. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E and $f : X \to X$ be a continuous function.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} = f(\hat{x})$.

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