

## Lacunary strong convergence with respect to a sequence of modulus functions

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*Abstract.* The definition of lacunary strong convergence is extended to a definition of lacunary strong convergence with respect to a sequence of modulus functions in a Banach space. We study some connections between lacunary statistical convergence and lacunary strong convergence with respect to a sequence of modulus functions in a Banach space.

*Keywords:* lacunary sequence, modulus function, statistical convergence, Banach space

*Classification:* 40A05, 40F05

### 1. Introduction

By a lacunary sequence  $\theta = (k_r)$  where  $k_0 = 0$ , we mean an increasing sequence of positive integers with  $h_r = k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ . The intervals determined by  $\theta$  will be denoted by  $I_r = (k_{r-1}, k_r]$  and the ratio  $k_r/k_{r-1}$  will be denoted by  $q_r$ . The sequence space of lacunary strongly convergent sequences  $N_\theta$  was defined by Freedman et al. [4], as follows:

$$N_\theta = \{x = (x_i) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} |x_i - l| = 0 \text{ for some } l\}.$$

Let  $\|x\|_\theta = \sup_r (h_r^{-1} \sum_{i \in I_r} |x_i|)$ , whenever  $x \in N_\theta$ . Then  $(N_\theta, \|\cdot\|_\theta)$  is a BK-space.  $N_\theta^0$  denotes the subset of all sequences which are lacunary strongly convergent to zero.  $(N_\theta^0, \|\cdot\|_\theta)$  is also a BK-space.

There is a strong connection between  $N_\theta$  and the sequence space  $|\sigma_1|$ , which is defined by

$$|\sigma_1| = \{x = (x_i) : \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n |x_i - l| = 0 \text{ for some } l\}.$$

In the special case  $\theta = (2^r)$ , we have  $N_\theta = |\sigma_1|$ .

The well known space  $\hat{c}$ , the space of all almost convergent sequences was defined by Lorentz [9]. Later  $[\hat{c}]$  the space of strong almost convergence was

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introduced by Maddox [10] and also independently by Freedman et al. [4]. This sequence space was defined as follows:

$$[\hat{c}] = \{x = (x_i) : \lim_{n \rightarrow \infty} n^{-1} \sum_{i=p+1}^{p+n} |x_i - l| = 0 \text{ uniformly in } p, \text{ for some } l\}.$$

We denote the space of all sequences which are strongly almost convergent to zero by  $[\hat{c}_0]$ . In [15], the spaces  $[\hat{c}_0]$  and  $[\hat{c}]$  were extended to  $[\hat{c}_0(f)]$  and  $[\hat{c}(f)]$ .

Let  $X$  be a Banach space. We define  $s(X)$  to be the vector space of all  $X$ -valued sequences,  $l_\infty(X)$  the vector space of all bounded  $X$ -valued sequences and  $c(X)$  the vector space of all convergent  $X$ -valued sequences. Thus  $x = (x_i) \in l_\infty(X)$ , if  $\sup \|x_i\| < \infty$ , where  $x_i \in X$  for  $i \in N$ . Consequently  $l_\infty(X)$  becomes a Banach space with the natural coordinatewise operations and  $\|x\| = \sup_i \|x_i\|$  for  $x \in l_\infty(X)$ .

The notion of a modulus function was introduced by Nakano [13]. We recall that a modulus  $f$  is a function from  $[0, \infty)$  to  $[0, \infty)$  such that (i)  $f(x) = 0$  if and only if  $x = 0$ , (ii)  $f(x + y) \leq f(x) + f(y)$  for  $x, y \geq 0$ , (iii)  $f$  is increasing and (iv)  $f$  is continuous from the right at 0. It follows that  $f$  must be continuous on  $[0, \infty)$ . Connor [2], Maddox [11], [12], Kolk [8], Pehlivan and Fisher [16] and Ruckle [19] used a modulus function to construct sequence spaces.

Now let  $S$  be the space of sequences of modulus functions  $F = (f_i)$  such that  $\lim_{u \rightarrow 0^+} \sup_i f_i(u) = 0$ . Throughout this paper the sequence of modulus functions determined by  $F$  will be denoted by  $F = (f_i) \in S$  for every  $i \in N$ .

The purpose of this paper is to introduce and study a concept of lacunary strong convergence with respect to a sequence of modulus functions in a Banach space.

## 2. Inclusion theorems

We now introduce the generalizations of the lacunary strongly convergent sequences and investigate some inclusion relations.

**Definition 2.1.** Let  $F = (f_i)$  be a sequence of modulus functions in  $S$ . Let  $X$  be a Banach space. We define the spaces

$$N_\theta(X) = \{x = (x_i) \in s(X) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} \|x_i - l\| = 0 \text{ for some } l \in X\},$$

$$N_\theta(X, F) = \{x = (x_i) \in s(X) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l\|) = 0 \text{ for some } l \in X\},$$

$$N_\theta^0(X, F) = \{x = (x_i) \in s(X) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} f_i(\|x_i\|) = 0\}.$$

$N_\theta(X)$ ,  $N_\theta(X, F)$  and  $N_\theta^0(X, F)$  are linear spaces. We consider only  $N_\theta(X, F)$ . Suppose that  $x_i \rightarrow l$  in  $N_\theta(X, F)$ ,  $y_i \rightarrow l'$  in  $N_\theta(X, F)$  and  $\alpha, \gamma$  are in  $C$ . Then

there exist integers  $K_\alpha$  and  $M_\gamma$  such that  $|\alpha| \leq K_\alpha$  and  $|\gamma| \leq M_\gamma$ . We have

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} f_i(\|\alpha x_i + \gamma y_i - (\alpha l + \gamma l')\|) \\ \leq K_\alpha h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l\|) + M_\gamma h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l'\|). \end{aligned}$$

This implies that  $\alpha x + \gamma y \rightarrow \alpha l + \gamma l'$  in  $N_\theta(X, F)$ . Note that if we put  $f_i = f$  for  $i \in N$  then  $N_\theta(X, F) = N_\theta(X, f)$ . We write  $N_\theta(X, f) = N_\theta(X)$  for  $f(x) = x$ .

**Proposition 2.2** ([16]). *Let  $f$  be a modulus and let  $0 < \delta < 1$ . Then for each  $\|u\| \geq \delta$ , we have  $f(\|u\|) \leq 2f(1)\delta^{-1}\|u\|$ .*

PROOF:

$$f(\|u\|) \leq f(1 + [\|u\|/\delta]) \leq f(1) + f([\|u\|/\delta]) \leq f(1)(1 + \|u\|/\delta) \leq 2f(1)\|u\|/\delta.$$

where  $[\|u\|/\delta]$  denotes the integer part of  $\|u\|/\delta$ . □

**Theorem 2.3.** *Let  $X$  be a Banach space and let  $F = (f_i)$  be a sequence of modulus functions in  $S$ . If  $x = (x_i)$  is lacunary strongly convergent to  $l$  in  $X$ , then  $x = (x_i)$  is lacunary strongly convergent to  $l$  in  $X$  with respect to  $F$ , i.e.  $N_\theta(X) \subset N_\theta(X, F)$ .*

PROOF: Let  $F = (f_i)$  be a sequence modulus functions in  $S$  and put  $\sup_i f_i(1) = M$ . Let  $x \in N_\theta(X)$ . Then we have

$$A_r(X) = h_r^{-1} \sum_{i \in I_r} \|x_i - l\| \rightarrow 0 \text{ as } r \rightarrow \infty, \text{ for some } l \in X.$$

Let  $\varepsilon > 0$  and choose  $\delta$  with  $0 < \delta < 1$  such that  $f_i(u) < \varepsilon$  ( $i \in N$ ) for every  $u$  with  $0 \leq u \leq \delta$ . We can write

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l\|) &= h_r^{-1} \sum_{\substack{i \in I_r \\ \|x_i - l\| \leq \delta}} f_i(\|x_i - l\|) + h_r^{-1} \sum_{\substack{i \in I_r \\ \|x_i - l\| > \delta}} f_i(\|x_i - l\|) \\ &\leq h_r^{-1}(h_r \varepsilon) + h_r^{-1} 2M\delta^{-1} h_r A_r(X), \end{aligned}$$

by Proposition 2.2. Letting  $r \rightarrow \infty$ , it follows that  $x \in N_\theta(X, F)$ . □

**Theorem 2.4.** *Let  $X$  be a Banach space and  $F = (f_i)$  be a sequence of modulus functions. If  $\lim_{u \rightarrow \infty} \inf_i f_i(u)/u > 0$ , then  $N_\theta(X, F) = N_\theta(X)$ .*

PROOF: If  $\lim_{u \rightarrow \infty} \inf_i f_i(u)/u > 0$  then there exists a number  $c > 0$  such that  $f_i(u) > cu$  for  $u > 0$  and  $i \in N$ . We have  $x \in N_\theta(X, F)$ . Clearly

$$h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l\|) \geq h_r^{-1} \sum_{i \in I_r} c\|x_i - l\| = ch_r^{-1} \sum_{i \in I_r} \|x_i - l\|,$$

therefore  $x \in N_\theta(X)$ . By using Theorem 2.3 the proof is complete.  $\square$

We now give an example to show that  $N_\theta(X, F) \neq N_\theta(X)$  in the case when  $\lim_{u \rightarrow \infty} \inf_i f_i(u)/u = 0$ . Consider the sequence  $f_i(x) = x^{1/(i+1)}$  ( $i \geq 1, x > 0$ ) of modulus functions. Now define  $x_i = h_r v$  if  $i = k_r$  for some  $r \geq 1$  and  $x_i = \theta$  otherwise, where  $v \in X$  and  $\|v\| = 1$ . This yields

$$h_r^{-1} \sum_{i \in I_r} f_i(\|x_i\|) = h_r^{-1}(f_{k_r}(h_r \|v\|)) = h_r^{-1} h_r^{1/(1+k_r)} \rightarrow 0 \text{ as } r \rightarrow \infty$$

and so  $x \in N_\theta(X, F)$ . But

$$h_r^{-1} \sum_{i \in I_r} \|x_i\| = h_r^{-1} h_r \|v\| \rightarrow 1 \text{ as } r \rightarrow \infty$$

and so  $x \notin N_\theta(X)$ .

**Proposition 2.5.** *If  $f_i = f$  for  $i \in N$ , then  $[\hat{c}(X, f)] \subset N_\theta(X, f)$  for every lacunary sequence  $\theta$ , where*

$[\hat{c}(X, f)] = \{x = (x_i) \in s(X) : \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n f(\|x_{i+p} - l\|) = 0, \text{ for some } l \in X \text{ uniformly in } p\}$ .

To show that  $N_\theta^0(X, f)$  strictly contains

$$[\hat{c}_0(X, f)] = \{x = (x_i) \in s(X) : \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n f(\|x_{i+p}\|) = 0 \text{ uniformly in } p\},$$

we proceed as in [4; p. 513]. We define  $x = (x_i)$  by  $x_i = v$  if  $k_{r-1} < i \leq k_{r-1} + [\sqrt{h_r}]$  for some  $r$  and  $x_i = \theta$  otherwise, where  $v \in X$  and  $\|v\| = 1$ . It follows that  $x \notin [\hat{c}_0(X, f)]$ . However  $x \in N_\theta^0(X, f)$  since

$$h_r^{-1} \sum_{i \in I_r} f(\|x_i\|) = h_r^{-1} [\sqrt{h_r}] f(1) \rightarrow 0 \text{ as } r \rightarrow \infty.$$

If  $f_i = f$  for  $i \in N$  we can show as in [4] that  $|\sigma_1(X, f)| = N_\theta(X, f)$  if and only if  $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ , where  $|\sigma_1(X, f)| = \{x = (x_i) \in s(X) : \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n f(\|x_i - l\|) = 0 \text{ for some } l \in X\}$ .

**Proposition 2.6.** *Let  $X$  be a Banach space. Let  $\theta = (k_r)$  be a lacunary sequence with  $\liminf_r q_r > 1$  then for any modulus  $f$ ,  $|\sigma_1(X, f)| \subset N_\theta(X, f)$ .*

PROOF: It is enough to show that  $|\sigma_1(X, f)|^0 \subset N_\theta^0(X, f)$ . Suppose  $\liminf_r q_r > 1$ . There exists  $\delta > 0$  such that  $q_r = (k_r/k_{r-1}) \geq 1 + \delta$  for sufficiently large  $r$ . We

have, for sufficiently large  $r$ , that  $(k_r/h_r) \leq (1 + \delta)/\delta$  and  $(h_r/k_r) \geq \delta/(1 + \delta)$ . Now write

$$\begin{aligned} k_r^{-1} \sum_{i=1}^{k_r} f(\|x_i\|) &\geq k_r^{-1} \sum_{i \in I_r} f(\|x_i\|) = (h_r/k_r) h_r^{-1} \sum_{i \in I_r} f(\|x_i\|) \\ &\geq (\delta/(1 + \delta)) h_r^{-1} \sum_{i \in I_r} f(\|x_i\|), \end{aligned}$$

from which we deduce that  $|\sigma_1(X, f)|^0 \subset N_\theta^0(X, f)$  for any modulus  $f$ .  $\square$

**Proposition 2.7.** *Let  $X$  be a Banach space. Let  $\theta = (k_r)$  be a lacunary sequence with  $\limsup_r q_r < \infty$  then for any modulus  $f$ ,  $N_\theta(X, f) \subset |\sigma_1(X, f)|$ .*

PROOF: Let  $x \in N_\theta^0(X, f)$  and  $\varepsilon > 0$ . There exists  $j_0$  such that for every  $j \geq j_0$

$$H_j = h_j^{-1} \sum_{i \in I_j} f(\|x_i\|) < \varepsilon.$$

We can also find  $M > 0$  such that  $H_j \leq M$  for all  $j$ . If  $\limsup_r q_r < \infty$  then there exists  $B > 0$  such that  $q_r < B$  for every  $r$ . Now let  $n$  be any integer with  $k_{r-1} < n \leq k_r$ . Then

$$\begin{aligned} n^{-1} \sum_{i=1}^n f(\|x_i\|) &\leq k_{r-1}^{-1} \sum_{i=1}^{k_r} f(\|x_i\|) = k_{r-1}^{-1} \left\{ \sum_{i \in I_1} f(\|x_i\|) + \dots + \sum_{i \in I_r} f(\|x_i\|) \right\} \\ &= k_{r-1}^{-1} \left\{ \sum_{j=1}^{j_0} \sum_{i \in I_j} f(\|x_i\|) + \sum_{j=j_0+1}^r \sum_{i \in I_j} f(\|x_i\|) \right\} \\ &\leq k_{r-1}^{-1} \sum_{j=1}^{j_0} \sum_{i \in I_j} f(\|x_i\|) + \varepsilon(k_r - k_{j_0}) k_{r-1}^{-1} \\ &= k_{r-1}^{-1} \{h_1 H_1 + h_2 H_2 + \dots + h_{j_0} H_{j_0}\} + \varepsilon(k_r - k_{j_0}) k_{r-1}^{-1} \\ &\leq k_{r-1}^{-1} \left( \sup_{1 \leq i \leq j_0} H_i \right) k_{j_0} + \varepsilon(k_r - k_{j_0}) k_{r-1}^{-1} < M k_{r-1}^{-1} k_{j_0} + \varepsilon B \end{aligned}$$

which yields that  $x \in |\sigma_1(X, f)|^0$ .  $\square$

The next result follows from Proposition 2.6 and 2.7.

**Theorem 2.8.** *Let  $\theta = (k_r)$  be a lacunary sequence with  $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ . Then  $|\sigma_1(X, f)| = N_\theta(X, f)$ . In particular we have  $N_{2r}(X, f) = |\sigma_1(X, f)|$ .*

### 3. Some results on $X$ -lacunary statistical convergence

We now introduce natural relationship between lacunary strong convergence with respect to a sequence of modulus functions in Banach space and lacunary statistical convergence in a Banach space. In [3], Fast introduced the idea of statistical convergence, which is closely related to the concept of natural density or asymptotic density of subsets of the positive integers  $N$ . These ideas were later studied in [1], [5], [17] and [18]. If  $K$  is a subset of the positive integers  $N$ , then  $K_n$  denotes the set  $\{k \in K : k \leq n\}$  and  $|K_n|$  denotes cardinality of  $K_n$ . The natural density of  $K$  is given by  $\delta(K) = \lim_{n \rightarrow \infty} n^{-1}|K_n|$ , see [14]. A sequence  $x = (x_i)$  is statistically convergent to  $l$  if for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} n^{-1}|K(\varepsilon)| = 0,$$

where  $K(\varepsilon) = \{i \in N : |x_i - l| \geq \varepsilon\}$  and  $|K(\varepsilon)|$  denotes cardinality of  $K(\varepsilon)$ . The set of all statistically convergent sequences is denoted by  $St$ .

Recently Fridy and Orhan [6], [7] introduced the following definition of lacunary statistical convergence.

**Definition 3.1.** Let  $\theta$  be a lacunary sequence. Then a sequence  $x = (x_i)$  is said to be lacunary statistically convergent to a number  $l$  if for every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow \infty} h_r^{-1}|K_\theta(\varepsilon)| = 0,$$

where  $K_\theta(\varepsilon) = \{i \in I_r : |x_i - l| \geq \varepsilon\}$ . The set of all lacunary statistically convergent sequences is denoted by  $St_\theta$ .

Some results on  $St_\theta$ -convergence and  $St$ -convergence were given in [7]. It was shown there that  $St = St_\theta$  if and only if  $1 < \lim_r \inf q_r \leq \lim_r \sup q_r < \infty$ .

**Definition 3.2.** Let  $\theta$  be a lacunary sequence. Then a sequence  $x = (x_i) \in s(X)$  is said to be  $X$ -lacunary statistically convergent to an  $l \in X$  if for every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow \infty} h_r^{-1}|\{i \in I_r : \|x_i - l\| \geq \varepsilon\}| = 0.$$

The set of all such sequences  $x = (x_i)$  is denoted by  $St_\theta(X)$ .

In the next section we establish inclusion relations between  $St_\theta(X)$  and  $N_\theta(X, F)$ .

**Theorem 3.3.** Let  $F = (f_i)$  be a sequence of modulus functions in  $S$ . Let  $X$  be a Banach space. Then  $N_\theta(X, F) \subset St_\theta(X)$  if and only if  $\inf_i f_i(u) > 0$ , ( $u > 0$ ).

PROOF: If  $\inf_i f_i(u) > 0$  then there exists a number  $\alpha > 0$  such that  $f_i(u) \geq \alpha$  for  $u > 0$  and  $i \in N$ . Let  $x \in N_\theta(X, F)$ ,  $\varepsilon > 0$  and  $K_\theta(X, \varepsilon) = \{i \in I_r : \|x_i - l\| \geq \varepsilon\}$  then

$$h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l\|) \geq h_r^{-1} \sum_{i \in K_\theta(X, \varepsilon)} f_i(\|x_i - l\|) \geq \alpha h_r^{-1} |K_\theta(X, \varepsilon)|$$

and it follows that  $x \in St_\theta(X)$ .

Conversely we can select subsequence  $k_{r_j}$  of the lacunary sequence and choose a number  $z \geq \varepsilon > 0$  such that  $f_i(z) = 0$  for  $i \in I_{r_j}$ . Now define a sequence  $x = (x_i)$  by putting  $x_i = zv$  if  $i \in I_{r_j}$  for some  $j = 1, 2, \dots$  and  $x_i = \theta$  otherwise, where  $v \in X$  and  $\|v\| = 1$ . Then we have  $x \in N_\theta(X, F)$  but  $x \notin St_\theta$ .

**Theorem 3.4.** *Let  $F = (f_i)$  be a sequence of modulus functions in  $S$ . Let  $X$  be a Banach space. Then  $St_\theta(X) \subset N_\theta(X, F)$  if and only if  $\sup_u \sup_i f_i(u) < \infty$ .*

PROOF: We suppose  $T(u) = \sup_i f_i(u)$  and  $T = \sup_u T(u)$ . Let  $x \in St_\theta(X)$ . Since  $f_i(u) \leq T$  for  $i \in N$  and  $u > 0$ , we have

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l\|) &= h_r^{-1} \left\{ \sum_{\substack{i \in I_r \\ \|x_i - l\| \geq \varepsilon}} f_i(\|x_i - l\|) + \sum_{\substack{i \in I_r \\ \|x_i - l\| < \varepsilon}} f_i(\|x_i - l\|) \right\} \\ &\leq h_r^{-1} \left\{ T |\{i \in I_r : \|x_i - l\| \geq \varepsilon\}| + h_r T(\varepsilon) \right\}. \end{aligned}$$

Taking the limit as  $\varepsilon \rightarrow 0$ , it follows that  $x \in N_\theta(X, F)$ , proving the sufficiency.

Conversely, suppose that  $\sup_u \sup_i f_i(u) = \infty$ . Then we have  $0 < u_1 < u_2 < \dots < u_{r-1} < u_r < \dots$  such that  $f_{k_r}(u_r) \geq h_r$  for  $r \geq 1$ . We define the sequence  $x = (x_i)$  by  $x_i = u_r v$  if  $i = k_r$  for some  $r = 1, 2, \dots$  and  $x_i = \theta$  otherwise, where  $v \in X$  and  $\|v\| = 1$ . We have  $x \in St_\theta(X)$  but  $x \notin N_\theta(X, F)$ .  $\square$

**Corollary 3.5.** *Let  $F = (f_i)$  be a sequence of modulus functions in  $S$  and let  $X$  be a Banach space. Then  $N_\theta(X, F) = St_\theta(X)$  if and only if  $\inf_i f_i > 0$  and  $\sup_u \sup_i f_i(u) < \infty$ . In particular, if  $f_i = f$  is a modulus function, we have  $N_\theta(X, f) = St_\theta(X)$  if and only if  $f$  is bounded.*

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