Singular quadratic functionals of one dependent variable

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Abstract. Singular quadratic functionals of one dependent variable with nonseparated boundary conditions are investigated. Necessary and sufficient conditions for nonnegativity of these functionals are derived using the concept of *coupled point* and *singularity condition*. The paper also includes two comparison theorems for coupled points with respect to the various boundary conditions.

Keywords: quadratic functional, singular quadratic functional, periodic and antiperiodic boundary condition, conjugate point, coupled point, singularity condition

Classification: 34C10, 49B10, 34A10

1. Introduction

The purpose of this paper is to study singular quadratic functionals of one dependent variable with periodic and antiperiodic boundary conditions. Our results, together with [3] and some modification of the results of [6], [7], [8], complete the analysis of the problem of nonnegativity of quadratic functionals in scalar case.

Let $r^{-1}(\cdot), p(\cdot) \in L_{loc}[a,\infty)$, $r(\cdot) > 0$, and $\alpha, \gamma \in \mathbf{R}$. We study the necessary and sufficient conditions for nonnegativity of the singular quadratic functional

$$\mathcal{I}(\eta; a, \infty) = \liminf_{b\to\infty} \mathcal{I}(\eta; a, b) = \tag{1.1}$$
$$\alpha\eta^2(a) + \liminf_{b\to\infty}\big[\gamma\eta^2(b) + \int_a^b r(s)\eta'^2(s) - p(s)\eta^2(s)\, ds\big],$$

over all functions $\eta \in W^{1,2}_{\text{loc}}[a,\infty)$ subject to the boundary condition

$$\lim_{b\to\infty} D\begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix} = 0, \tag{1.2}$$

where D is a 2×2 matrix.

Singular quadratic functionals have been studied for the first time by Leighton and Morse [6] and followed by the subsequent papers [7], [8], [10], [11], [12] in the special case $D = I$ — the identity matrix, i.e. for functions satisfying zero boundary conditions. Note that in [6], [7], [8] the singularity is considered at the point $t = 0$, but the change of independent variable $t \to \frac{1}{t}$ brings this singularity

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to ∞ as considered here. The term *singular* functional means that the functions p, r^{-1} may fail to be Lebesgue integrable in the interval under consideration.

It was shown that $I(\eta; a, \infty) \geq 0$ for the functions satisfying $\eta(a) = 0 = \lim_{b\to\infty} \eta(b)$ if and only if the associated Euler equation

$$(r(t)y')' + p(t)y = 0 \tag{1.3}$$

is disconjugate on $[a, \infty)$ and the so-called singularity condition holds:

$$\liminf_{t\to\infty} \eta^2(t)\frac{r(t)y_a'(t)}{y_a(t)} \geq 0 \tag{1.4}$$

for any admissible function η for which $I(\eta; a, \infty) < \infty$, where y_a is the solution of (1.3) for which $y_a(a) = 0,\ y_a'(a) \neq 0$.

The quadratic functionals of n-dependent variable on a compact interval over functions satisfying the general boundary condition

$$D\begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix} = 0 \tag{1.5}$$

where D is a $2n \times 2n$ matrix, have been studied by Zeidan and Zezza [13] and the concepts of the so-called *coupled point* and *regularity condition* were introduced here. It was shown, see [13], [14], [15], that the nonexistence of a point $c \in [a, b)$ coupled with a together with the regularity condition yields a necessary and sufficient condition for the nonnegativity of the quadratic functional. However, for singular functionals the analogous ideas cannot be applied for proving necessary as well as sufficient condition. For this reason in [3] the coupled point relative to quadratic functionals of scalar variable with free and periodic boundary condition was described by means of a solution of the Riccati equation associated with (1.3)

$$w' + r^{-1}(t)w^2 + p(t) = 0. \tag{1.6}$$

It was shown that for free end points, similarly as for zero end points, nonexistence of a coupled point with a is no longer sufficient for nonnegativity of the functional and the "free end points" analogy of the classical singularity condition (1.4) was introduced.

Here we continue this research. We introduce the concept of the *singularity condition* for functionals with periodic and antiperiodic boundary condition and we show that nonexistence of a coupled point together with validity of the regularity and singularity condition yield necessary and sufficient condition for nonnegativity of the functional under consideration. The used method enables to improve the result on ordering of coupled point with respect to the various boundary condition on a compact interval and to prove similar result for singular functionals.

The boundary conditions (1.2) can be transformed by a transformation preserving nonnegativity of a quadratic functional (see Remark 5) into one of the following cases:

I. rank $D = 0$ .... *free end points*
II. rank $D = 1$
II (a) $\eta(a) = -\lim_{b\to\infty} \eta(b)$ .... *antiperiodic boundary conditions*
II (b) $\eta(a) = \lim_{b\to\infty} \eta(b)$ .... *periodic boundary conditions*
II (c) $\eta(a) = 0$ .... *free right (singular) end point*
II (d) $\lim_{b\to\infty} \eta(b) = 0$ .... *free left (regular) end point*
III. rank $D = 2$ .... *zero boundary conditions*

The paper is organized as follows. In Section 2 there are presented our main results on nonnegativity of singular quadratic functionals of one dependent variable and two theorems about ordering of coupled points with respect to the various boundary conditions. Section 3 contains auxiliary results; the main result of this section is description of coupled point in terms of Riccati equation. The three theorems from Section 2 are proved in Section 4 and the last section contains two examples illustrating our results.

2. Main results

We start with two definitions, which are analogical to the compact interval case given in [13].

Definition 1. The functional $\mathcal{I}(\eta; a, \infty)$ is said to satisfy the *regularity condition* if $\mathcal{I}(\eta; a, \infty) \ge 0$ for any constant function η if this is admissible.

That is, if the constant function is admissible, then

$$\mathbf{k} := \alpha + \gamma - \limsup_{b\to\infty} \int_a^b p(s)\, ds \ge 0. \tag{2.1}$$

Definition 2. A point $c \in [a, \infty)$ is said to be the *coupled point* with a relative to $\mathcal{I}(\eta; a, \infty)$ if there exists a nontrivial solution y of (1.3) and $\sigma \in \mathbb{R}^2$ such that

$$y(t) \not\equiv y(c) \text{ on } (c, \infty) \qquad D\begin{pmatrix} y(a) \\ y(c) \end{pmatrix} = 0,$$
$$\left[\begin{pmatrix} \alpha y(a) \\ \gamma y(c) \end{pmatrix} + \begin{pmatrix} -r(a)y'(a) \\ r(c)y'(c) \end{pmatrix} - \limsup_{b\to\infty} \begin{pmatrix} 0 \\ y(c)\int_c^b p \end{pmatrix}\right] = D^T\sigma.$$

If the interval $[a, c)$ does not contain any other coupled point with a, then c is said to be the *first coupled point*.

Recall that replacing $\limsup_{b\to\infty} \int_a^b p(s)\, ds$ by $\int_a^b p(s)\, ds$ and $\limsup_{b\to\infty} \int_c^b p(s)\, ds$ by $\int_c^b p(s)\, ds$ in Definition 1 and 2, respectively, we have definitions of regularity condition and coupled point for functionals on a compact interval $[a, b]$ introduced in [13].

Remark 1. The point a is coupled with a if and only if $y \equiv \text{const}$ is admissible, it is not an extremal on $[a, \infty)$ and $\mathbf{k} = 0$.

The first theorem gives necessary and sufficient conditions for nonnegativity of singular quadratic functionals.

Theorem 1. *$\mathcal{I}(\eta; a, \infty) \geq 0$ if and only if the regularity condition (2.1) holds, there exists no coupled point $c \in [a, \infty)$ with a and the singularity condition is satisfied: for any admissible function η for which $I(\eta; a, \infty) < \infty$ and for the solution w of (1.6) the following holds*

Case I (free end points)

$$\liminf_{t \to \infty} \eta^2(t)[w(t) + \gamma] \geq 0, \quad w(a) = \alpha.$$

Case II (a) (antiperiodic boundary conditions)

$$\liminf_{t \to \infty} \left[\eta^2(a)[\alpha - w(a)] + [\gamma + w(t)]\eta^2(t) + \frac{(\eta(t) - \eta(a))^2}{u(t)} \right] \geq 0$$

where $\int_a^t r^{-1} w \, ds = 0$ and u is the solution of (1.3) *for which $u(a) = 0$, $r(a)u'(a) = 1$.*

Case II (b) (periodic boundary conditions)

$$\liminf_{t \to \infty} \left[\eta^2(a)[\alpha - w(a)] + [\gamma + w(t)]\eta^2(t) + \frac{(\eta(t) - \eta(a))^2}{u(t)} \right] \geq 0$$

where w and u is the same as in Case II (a).

Case II (c) (free singular end point)

$$\liminf_{t \to \infty} \eta^2(t)[w(t) + \gamma] \geq 0, \quad w(a+) = \infty.$$

Case II (d) (free regular end point)

$$\liminf_{t \to \infty} \eta^2(t) w(t) \geq 0, \quad w(a) = \alpha.$$

Case III (zero boundary conditions)

$$\liminf_{t \to \infty} \eta^2(t) w(t) \geq 0, \quad w(a+) = \infty.$$

In the next remark we give some comments concerning Theorem 1. All statements presented in this remark are proved in all details in Section 3.

Remark 2. (i) If we consider the quadratic functional on a compact interval $[a, b]$ then nonexistence of a point $c \in [a, b)$ coupled with a and regularity condition imply that singularity condition is satisfied (this means that replacing $\liminf_{t \to \infty}$ by $\liminf_{t \to b}$ all lower limits are nonnegative). From this point of view, Theorem 1 presents an alternative proof of the statement that $\mathcal{I}(\eta; a, b)$ is nonnegative if and only if the regularity conditions hold and there is no point $c \in [a, b)$ coupled with a.

(ii) A function η satisfying the zero boundary condition $\eta(a) = 0 = \lim_{b\to\infty} \eta(b)$ is admissible for all types of boundary conditions. For this class of functions all singularity conditions I, II (a)–(d) reduce to the classical singularity condition III.

(iii) The validity of the regularity condition (2.1) in Cases I, II (a), (c), (d) implies

$$\limsup_{b\to\infty} \int_a^b p < \infty \tag{2.2}$$

since constant functions are admissible. In the antiperiodic Case II (b) constants are not admissible, but if (2.2) is not satisfied then for the function

$$\eta(t) = \begin{cases} 1 + a - t, & t \in [a, a+2] \\ -1 & t > a+2 \end{cases}$$

the functional attains again the value $-\infty$.

In the following, we give two comparison theorems for coupled points; the first one concerns the regular functionals $\mathcal{I}(\eta, a, b)$, the second one the singular functionals $\mathcal{I}(\eta, a, \infty)$. In [4] it is proved for regular functionals $\mathcal{I}(\eta, a, b)$ by index theory method the ordering of coupled points with respect to the various boundary conditions (1.5) in terms "$\leq$" but the general character of this theory does not enable to distinguish cases "$=$" and "$<$". Our method enables to improve this result. In the following theorem we use the same classification of boundary conditions (1.5) on $[a, b]$ as the one on $[a, \infty)$.

Theorem 2. *Let c_f, c_p, c_a, c_0 be first coupled points with a relative to $\mathcal{I}(\eta, a, b)$ with free end points, periodic, antiperiodic, zero boundary conditions, respectively. Then it holds*

$$c_f \leq c_p < c_a \leq c_0.$$

Moreover, let u, v be the solutions of (1.3) *satisfying $u(a) = 0 = v'(a)$, $r(a)u'(a) = 1 = v(a)$. If*

$$\alpha u(t) + v(t) - 1 \neq 0, \quad t \in [a, c_f]$$

then $c_f < c_p$. If

$$\int^{c_0} \frac{r(t)u'(t) + 1)^2}{r(t)u^2(t)}\, dt = \infty$$

then $c_a < c_0$.

A similar statement holds also for functionals on a noncompact interval. Before stating this, we need to define the conjugate point of ∞.

Definition 3. Let u be the solution of (1.3) such that $u(a) = 0$, $r(a)u'(a) = 1$. We say that ∞ is the *generalized first conjugate point* of a if $u(t) > 0$ for $t \in (a, \infty)$ and

$$\int^{\infty} \frac{dt}{r(t)u^2(t)} = \infty. \tag{2.3}$$

Recall that solution satisfying this condition is said to be *principal* at ∞ and if b is conjugate with a then trivially $\int^b \frac{dt}{r(t)u^2(t)} = \infty$.

Theorem 3. *Let*

$$\limsup_{b\to\infty} \int_a^b p > -\infty. \tag{2.4}$$

Suppose that ∞ *is the generalized first conjugate point with* a *and the boundary condition* (1.2) *is one of the types* I, II (b), II (c), II (d), III. *Then there exists* $c \in [a,\infty)$ *which is coupled with* a *relative to* $\mathcal{I}(\eta; a, \infty)$.

Concerning the antiperiodic boundary condition II (a), *if*

$$\int^{\infty} \frac{(r(t)u'(t)+1)^2}{r(t)u^2(t)}\, dt = \infty, \tag{2.5}$$

then there exists $c \in [a,\infty)$ *coupled with* a *relative to* $\mathcal{I}(\eta; a, \infty)$. *Here* u *is the solution of* (1.3) *satisfying* (2.3).

Remark 3. If (2.4) is not satisfied then there is no point $c \in [a,\infty)$ coupled with a for any boundary condition I, II, see Lemma 7 and Remark 7.

At the end of this section we show that the boundary conditions of type I–III cover, up to a suitable transformation, all possible cases of boundary conditions (1.5). The same holds for boundary conditions (1.2). The cases rank $D = 0$ and rank $D = 2$ are trivial. Let rank $D = 1$ and

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Then at least one of the entries d_{ij}, $i, j = 1, 2$ is nonzero and the rows of D are linearly dependent, so it is sufficient to consider the first row. The cases $d_{11} = 0$ and $d_{12} = 0$ are equivalent to II (d) and II (c), respectively. If $d_{11} \neq 0$ and $d_{12} \neq 0$, the transformation

$$\eta(t) = h(t)u(t)$$

where $h(t) \neq 0$, transforms $\mathcal{I}(\eta; a, b)$ into the functional

$$[\alpha h^2(a) - r(a)h'(a)h(a)]u^2(a) + [\gamma h^2(b) + r(b)h'(b)h(b)]u^2(b) + \int_a^b [rh^2u'^2 - h((rh')' - ph)u^2],$$

see, e.g. [2], and the boundary condition (1.5) yields

$$d_{11}h(a)u(a) + d_{12}h(b)u(b) = 0.$$

Now, taking $h(t) \neq 0$ such that $h(a) = \frac{1}{|d_{11}|}$, $h(b) = \frac{1}{|d_{12}|}$, we see that the case $d_{11}d_{12} > 0$ can be reduced to the periodic boundary condition and the case $d_{11}d_{12} < 0$ to the antiperiodic one.

3. Auxiliary results: coupled point in terms of the Riccati equation

The idea of the Riccati equation method consists in the following: the quadratic functional is expressed by the Picone identity and by means of this expression an auxiliary function F is defined in order to describe a coupled point. More precisely, the point coupled with a is the point where the function F equals the constant $\mathbf{k}$ from regularity condition. The crucial property of F is its monotonicity.

Since the main subject of this paper are periodic and antiperiodic boundary conditions, we specify explicitly these two cases. Let

$$\mathcal{A}(\eta; a, \infty) = (\alpha + \gamma)\eta^2(a) + \liminf_{b\to\infty} \int_a^b [r(s)\eta'^2(s) - p(s)\eta^2(s)]\, ds,$$

over all $\eta \in W^{1,2}_{\text{loc}}[a, \infty)$ such that $\eta(a) = -\lim_{t\to\infty} \eta(t)$;
and

$$\mathcal{P}(\eta; a, \infty) = (\alpha + \gamma)\eta^2(a) + \liminf_{b\to\infty} \int_a^b [r(s)\eta'^2(s) - p(s)\eta^2(s)]\, ds,$$

over all $\eta \in W^{1,2}_{\text{loc}}[a, \infty)$ such that $\eta(a) = \lim_{t\to\infty} \eta(t)$. The definition of coupled point in these cases reads as follows:

A point $c \in [a, \infty)$ is said to be the *coupled point* with a relative to $\mathcal{A}(\eta; a, \infty)$ if there exists a nontrivial extremal y for which $y(a) = -y(c)$,

$$(3.1) \qquad (\alpha + \gamma)y(a) - r(c)y'(c) - r(a)y'(a) + y(a) \limsup_{b\to\infty} \int_c^b p = 0$$

and $y(t) \not\equiv y(c)$ on $[c, \infty)$.

A point $c \in [a, \infty)$ is said to be the *coupled point* with a relative to $\mathcal{P}(\eta; a, \infty)$ if there exists a nontrivial extremal y for which $y(a) = y(c)$,

$$(\alpha + \gamma)y(a) + r(c)y'(c) - r(a)y'(a) - y(a) \limsup_{b\to\infty} \int_c^b p = 0,$$

and $y(t) \not\equiv y(c)$ on $[c, \infty)$.

Again, replacing in the above definitions $\limsup_{b\to\infty} \int_c^b p$ by $\int_c^b p$, we have the definition of the coupled point for the functionals $\mathcal{A}(\eta; a, b)$, $\mathcal{P}(\eta; a, b)$ on a compact interval.

To introduce the auxiliary function F, we recall first some known results.

Lemma 1 ([1, p. 2]). *Let* (1.3) *be disconjugate on* $[a, b]$, $c \in (a, b]$, $\alpha, \beta \in \mathbf{R}$ *be arbitrary. Then there exists the unique solution* y *of* (1.3) *for which* $y(a) = \alpha$, $y(c) = \beta$.

Lemma 2. *Let* (1.3) *be disconjugate on* $[a,b)$, $c \in [a,b)$, $\eta \in W^{1,2}[a,b]$ *be such that* $\eta(a) = -\eta(c) \neq 0$ *and* $\eta(t) \equiv \eta(c)$ *for* $t \in [c,b]$. *If* w *is any solution of* (1.6) *which exists on* $[a,c]$, *then*

$$\mathcal{A}(\eta; a,b) = \eta^2(a)\left[(\alpha + \gamma - \int_a^b p) - (\int_a^c r^{-1}w^2 - \frac{1}{\eta^2(a)}\int_a^c r^{-1}(r\eta' - w\eta)^2)\right].$$

Proof: The statement follows from the well known Picone identity (c.f. [9, p. 73])

$$\begin{aligned}\mathcal{A}(\eta; a,b) &= (\alpha+\gamma)\eta^2(a) + \eta^2 w|_a^c + \int_a^c r^{-1}(r\eta' - w\eta)^2 - \eta^2(c)\int_c^b p \\ &= \eta^2(a)[\alpha + \gamma + w(c) - w(a) - \int_c^b p] + \int_a^c r^{-1}(r\eta' - w\eta)^2 \\ &= \eta^2(a)[\alpha+\gamma - \int_a^b p - \int_a^c r^{-1}w^2] + \int_a^c r^{-1}(r\eta' - w\eta)^2.\end{aligned}$$

□

The statement of Lemma 2 suggests the form of the function F in case of antiperiodic boundary condition (this is the part of the last expression depending on c)

$$F(c) = \int_a^c r^{-1}w^2 - \frac{1}{y^2(a)}\int_a^c r^{-1}(ry' - wy)^2. \tag{3.2}$$

The problem is *which* solution w of Riccati equation should be taken in (3.2) and we solve this in the following lemmata.

Lemma 3 ([3, Lemma 5]). *If* (1.3) *is disconjugate on* $[a,b)$ *then for every* $c \in (a,b)$ *there exists a solution* $w_c(t)$ *of* (1.6) *for which*

$$\int_a^c r^{-1}(t)w_c(t)\,dt = 0. \tag{3.3}$$

Remark 4. If y is a nonzero extremal satisfying $y(a) = y(c)$, then the corresponding solution $w_c = r^{-1}y'y^{-1}$ of (1.6) satisfies (3.3). Thus we have $\int_a^c r^{-1}(ry' - wy)^2 = 0$, i.e.,

$$\mathcal{P}(y; a,b) = y^2(a)[\alpha+\gamma - \int_a^b p - \int_a^c r^{-1}w^2] \tag{3.4}$$

and hence for periodic boundary condition the function F takes the form

$$F(c) = \int_a^c r^{-1}w^2.$$

In contrast to this case, a nontrivial extremal y satisfying antiperiodic condition $y(a) = -y(c)$ defines *no* solution of Riccati equation (1.6) which exists on the whole interval (a, c). For this reason, the method of investigation of coupled points in antiperiodic case in terms of the Riccati equation is more complicated than in periodic case.

The following lemma describes a certain extremal property of solution w of (1.6) satisfying (3.3).

Lemma 4. *Let* (1.3) *be disconjugate on* $[a, b)$, $c \in (a, b)$. *Let* y *be a nontrivial extremal satisfying*

$$y(a) = -y(c)$$

and let w_c *be the solution of* (1.6) *satisfying* (2.3). *Then*

$$\int_a^c r^{-1}(ry' - w_c y)^2 \leq \int_a^c r^{-1}(ry' - wy)^2$$

for any other solution w *of* (1.6) *which exists on* $[a, c]$. *Moreover, the minimal value is*

$$\int_a^c r^{-1}(ry' - w_c y)^2 = \frac{4y^2(a)}{u(c)},$$

where u *is the solution of* (1.3) *for which* $u(a) = 0$, $r(a)u'(a) = 1$.

Proof: Let u be the solution of (1.3) for which $u(a) = 0$, $r(a)u'(a) = 1$. The solutions u, y, are linearly independent, hence any solution w of (1.6) is of the form

$$w(t) = \frac{r(t)(Au'(t) + By'(t))}{Au(t) + By(t)},$$

A, B, being real constants. Since $B \neq 0$ for solutions which exist on $[a, c]$, we may suppose $B = 1$. Denote $z = Au + y$, $G(A) = \int_a^c r^{-1}(ry' - wy)^2$, where $w = rz'z^{-1}$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}A} G(A) &= \frac{\mathrm{d}}{\mathrm{d}A} \int_a^c r^{-1}(ry' - \frac{r(Au' + y')}{Au + y} y)^2 \\
&= \omega \frac{\mathrm{d}}{\mathrm{d}A} \left[A \int_a^c \frac{y'(Au + y) - y(Au' + y')}{(Au + y)^2} \right] \\
&= \omega \frac{\mathrm{d}}{\mathrm{d}A} \frac{Ay}{Au + y} \Big|_a^c = \omega \frac{\mathrm{d}}{\mathrm{d}A} \left[\frac{Ay(c)}{Au(c) + y(c)} - A \right] \\
&= \omega \left[\frac{y^2(c)}{(Au(c) + y(c))^2} - 1 \right]
\end{aligned}$$

where $\omega = r(y'u - yu')$ is the Wronskian of u and y. For $A_0 = -2y(c)u^{-1}(c) = 2y(a)u^{-1}(c)$ we have $\frac{\mathrm{d}}{\mathrm{d}A} G(A_0) = 0$ and the corresponding solution z of (1.3) satisfies $z(a) = z(c)$. Hence $w_c(t) = rz'z^{-1}$ satisfies $\int_a^c r^{-1}(t) w_c(t)\, dt = 0$ and the minimal value is $G(A_0) = 4y^2(a)u^{-1}(c)$. □

Lemma 5. *Let w_c and u be the same as in Lemmas 3 and 4. Then*

$$\frac{\mathrm{d}}{\mathrm{d}c}\int_a^c r^{-1}(t)w_c^2(t)\,dt = \frac{1}{r(c)}\left(\frac{r(c)u'(c)-1}{u(c)}\right)^2.$$

PROOF: Let $y_c(t)$ be the solution of (1.3) for which $y_c(a) = y_c(c)$. Then

$$r(t)\frac{\partial}{\partial c}y_c'(t)y_c(t) - r(t)y_c'(t)\frac{\partial}{\partial c}y_c(t) = \varrho = \text{const.}$$

Indeed, let u, v be the solutions of (1.3) for which $u(a) = 0$, $r(a)u'(a) = 1$, $v(c) = 0$, $r(c)v'(c) = -1$. Then $y_c(t) = v(a)u(t) + u(c)v(t)$ and

$$\begin{aligned}\varrho &= r(t)u'(c)v'(t)[v(a)u(t)+u(c)v(t)] - r(t)u'(c)v(t)[v(a)u'(t)+u(c)v'(t)]\\ &= u'(c)v(a)[r(t)v'(t)u(t) - r(t)u'(t)v(t)] = -u'(c)v^2(a).\end{aligned}$$

Now, let $w_c(t) = r(t)y_c'(t)y_c^{-1}(t)$. Then $\frac{\partial}{\partial c}w_c(t) = \varrho y_c^{-2}(t)$ and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}c}\int_a^c r^{-1}(t)w_c^2(t)\,dt = r^{-1}(c)w_c^2(c) + 2\int_a^c w_c(t)\frac{\partial}{\partial c}w_c(t)\,dt\\ &= r^{-1}(c)w_c^2(c) + 2\varrho\int_a^c \frac{r(t)y_c'(t)}{y_c^3(t)}\,dt\\ &= r^{-1}(c)\left(\frac{v(a)r(c)u'(c)-u(c)}{v(a)u(c)}\right)^2 - \varrho\frac{1}{y_c^2(t)}\Big|_a^c = \frac{1}{r(c)}\left(\frac{r(c)u'(c)-1}{u(c)}\right)^2.\end{aligned}$$

since the wronskian identity $r(u'v-v'u)|_{t=a} = r(u'v-v'u)|_{t=c}$ gives $u(c) = v(a)$. □

Lemma 6. *Let (1.3) be disconjugate on $[a,b)$, w_c and u be the same as in Lemma 3 and Lemma 4. Then the function F*

$$F(c) = \int_a^c r^{-1}w^2 - \frac{4}{u(c)}$$

is nondecreasing on $[a,b)$ and

$$\lim_{c\to a+} F(c) = -\infty. \tag{3.5}$$

Moreover, if b is a conjugate point of a and $r(b)u'(b) \neq -1$ then

$$\lim_{c\to b-} F(c) = \infty.$$

Proof: By previous lemmas

$$\frac{\mathrm{d}F}{\mathrm{d}c} = \frac{1}{r(c)}\left(\frac{r(c)u'(c)-1}{u(c)}\right)^2 + \frac{4u'(c)}{u^2(c)} = \frac{1}{r(c)}\left(\frac{r(c)u'(c)+1}{u(c)}\right)^2 \geq 0. \tag{3.6}$$

Since $\lim_{c\to a+}\int_a^c r^{-1}w_c^2 = 0$, (3.5) follows from Lemma 5. If b is a conjugate point of a then $\int^b r^{-1}u^{-2} = \infty$ and if $r(b)u'(b) \neq -1$ there exists $k > 0$ and $\tilde{c} \in (a,b)$ such that $(r(t)u'(t)+1)^2 > k$ for $t \in (\tilde{c}, b)$, hence

$$F(c) = F(\tilde{c}) + \int_{\tilde{c}}^c F'(t)\,dt \leq F(\tilde{c}) + k\int_{\tilde{c}}^c r^{-1}u^{-2}\,dt \to \infty \qquad \text{as } c \to b- .$$

□

Remark 5. Observe that Lemma 6 holds for *any* solution w of (1.6) which exists on $[a,c]$. The solution w_c has been taken because of its extremal property (Lemma 2.4) and since the computations concerning the function $F(c)$ are in this case simpler.

Remark 6. (i) In case of the periodic boundary condition the first coupled point of a may be described by means of a solution of Riccati equation (1.6) only under the so-called regularity assumption

$$\alpha + \gamma - \int_a^b p \geq 0$$

which means that the functional is nonnegative for nontrivial constant functions. In antiperiodic case we do not need a similar assumption since nontrivial constant functions are not admissible. Hence, in this case a cannot be its own coupled point; this fact reflects the relation (2.4).

(ii) In contrast to periodic boundary condition $y(a) = y(b)$ (see [3, Lemma 3]), in the case of the antiperiodic boundary condition it may happen that the coupled point with a relative to $\mathcal{A}(\eta; a, b)$ coincides with the first right conjugate point of a as shows Example 2.

We conclude this section with the "Riccati equation" description of the first coupled point relative to the functionals on a compact interval for all cases of boundary conditions I–III.

Lemma 7. *Let $\mathcal{I}(\eta; a, b)$ satisfy the regularity condition. A point $c \in [a, b)$ is the first coupled point with a relative to $\mathcal{I}(\eta; a, b)$ if and only if*

Case I. *The solution w of* (1.6) *such that $w(a) = \alpha$ satisfies*

$$\int_a^c r^{-1}w^2 = \alpha + \gamma - \int_a^b p$$

and there exists $d \in (c, b)$ *such that* $\int_c^d r^{-1}w^2 > 0$.

Case II (a). *The solution* w *from Lemma* 3 *and* u *from Lemma* 4 *satisfy*

$$F(c) := \int_a^c r^{-1}w^2 - \frac{4}{u(c)} = \alpha + \gamma - \int_a^b p$$

and there exists $d \in (c, b]$ *such that* $F(d) > F(c)$.

Case II (b). *The solution* w *from Lemma* 3 *satisfies*

$$\int_a^c r^{-1}w^2 = \alpha + \gamma - \int_a^b p$$

and there exists $d \in (c, b)$ *such that* $\int_c^d r^{-1}w^2 > 0$.

Case II (c). *For the solution* w *of* (1.6) *for which* $w(a+) = \infty$ *and for some* $\bar{a} \in (a, c)$ *the following holds:*

$$\int_{\bar{a}}^c r^{-1}w^2 = \gamma - \int_{\bar{a}}^b p + w(\bar{a})$$

and there exists $d \in (c, b)$ *such that* $\int_c^d r^{-1}w^2 > 0$.

Case II (d). *The solution* w *of* (1.6) *such that* $w(a) = \alpha$ *satisfies* $|w(c-)| = \infty$.

Case III. *The solution* w *of* (1.6) *such that* $w(a+) = \infty$ *exists on* (a, c) *and* $w(c-) = -\infty$.

PROOF: Case I. See [3, Lemma 2].

Case II (a). Let $c \in (a, b)$ be coupled with a and let y be the extremal which realizes this coupled point. By Lemma 2 we get

$$\mathcal{A}(\eta; a, b) = \eta^2(a)[\alpha + \gamma - \int_a^b p - F(c)] = 0\,.$$

Since $y(t) = \text{const}$ is not extremal on $[c, b]$, we have $p(t) \not\equiv 0$ on this interval and this implies $r(t)u'(t) \not\equiv -1$ for $t \in [c, b]$ and thus $F(\cdot)$ is not constant on $[c, b]$.

Conversely, if (2.5) holds and there exists $d \in (c, b]$ such that $F(d) > F(c)$ then according to (3.6) $ru' \not\equiv -1$ on $[c, b]$, i.e. $p(t) \not\equiv 0$ on this interval.

It is not difficult to verify that the function

$$\eta(t) = \begin{cases} y(t), & t \in [a, c] \\ y(c), & t \in [c, b], \end{cases}$$

where y is a nontrivial extremal satisfying $y(a) = -y(c)$, realizes the coupled point with a relative to $\mathcal{A}(\eta; a, b)$.

Case II (b). See [3, Lemma 4].

Case II (c). Let y be a nontrivial extremal which realizes the coupled point c, i.e. $y(a) = 0$ and $r(c)y'(c) + \gamma y(c) - y(c)\int_c^b p = 0$. If $y(c) = 0$ then $r(c)y'(c) = 0$, a contradiction, because y is nontrivial. Hence,

$$r(c)y'(c)y^{-1}(c) - \int_c^b p + \gamma = w(c) + \gamma - \int_c^b p = 0.$$

Integrating (1.6) from $\bar{a}$ to c and substituting for $w(c)$ we get the conclusion. As $y(t) \equiv y(c)$ is not an extremal on $[c, b)$, clearly there exists $d \in (c, b)$ such that $\int_c^d r^{-1}w^2 > 0$.

Case II (d) and III. The statements follow immediately from the definition of coupled and conjugate points. □

Remark 7. To obtain the Riccati equation characterization of coupled point for $\mathcal{I}(\eta, a, \infty)$ it is sufficient to replace $\int_a^b p$ by $\limsup_{b\to\infty}\int_a^b p$ in Lemma 7.

4. Proofs of main results

Proof of Theorem 1. Case I — see [3, Theorem 2].

Case II (a). (i) If there exists no point $c \in (a, \infty)$ coupled with a then (1.3) is disconjugate on (a, ∞). Indeed, suppose that there exists $b \in (a, \infty)$ which is conjugate with a. If $r(b)u'(b) + 1 = 0$, where u is the solution of (1.3) satisfying $u(a) = 0$, $r(a)u'(a) = 1$, then by Definition 3.2 b is coupled with a. If $r(b)u'(b) + 1 \neq 0$, then Lemma 6 implies $F(b-) = \infty$, hence by Lemma 7 there exists $c \in (a, b)$ which is coupled with a, a contradiction.

Let $\eta \in W^{1,2}_{loc}$ be any admissible function, $b > a$ is arbitrary and y is the extremal for which $y(a) = \eta(a)$, $y(b) = \eta(b)$. Disconjugacy of (1.3) on (a, ∞) implies

$$\int_a^b (r\eta'^2 - p\eta^2)\, dt \geq \int_a^b (ry'^2 - py^2)\, dt,$$

see [9, p. 71]. This inequality and Lemma 2 (with the solution $w_b(t)$ of (1.6) satisfying $\int_a^b r^{-1}w_b = 0$) give

$$\mathcal{A}(\eta; a, b) \geq \mathcal{A}(y; a, b) = y^2(a)(\alpha - w_b(a)) + y^2(b)[\gamma + w_b^2(b)] + \int_a^b r^{-1}(ry' - w_b y)^2.$$

If $\eta(a) \neq 0$, let us denote by v the solution of (1.3) for which $v(a) = \eta(a) = v(b)$ (i.e. $w_b = rv'v^{-1}$). Then $y(t) = (\eta(b) - \eta(a))u^{-1}(b)u(t) + v(t)$ and similarly as in the proof of Lemma 4 we have

$$\int_a^b r^{-1}(ry' - wy)^2 = \frac{(\eta(b) - \eta(a))^2}{u(b)}.$$

It follows

$$\mathcal{A}(\eta;a,b) \ge \mathcal{A}(y;a,b) = \eta^2(a)\gamma + w_b(b)\eta^2(b) - w_b(a)\eta^2(a) + \frac{(\eta(b)-\eta(a))^2}{u(b)},$$

hence, if (3.4) holds, $\mathcal{A}(\eta;a,\infty) = \liminf_{b\to\infty}\mathcal{A}(\eta;a,b) \ge 0$.

If $\eta(a)=0$, then $y(t)=u(t)\eta(b)u^{-1}(b)$ and one may directly verify that

$$\int_a^b r^{-1}(ry'-wy)^2 = \eta^2(b)u^{-1}(b).$$

Hence also in this case, $\mathcal{A}(\eta;a,b) = \liminf_{b\to\infty}\mathcal{A}(\eta;a,b) \ge 0$ provided (3.4) holds.

(ii) Now suppose that $\mathcal{A}(\eta;a,\infty) \ge 0$. That means that $\mathcal{A}(\eta;a,\infty)\ge 0$ also for any η for which $\eta(a)=0=\eta(\infty)$, i.e. (1.3) is disconjugate on $[a,\infty)$ by [6]. Suppose that there exists a point $c\in(a,\infty)$ coupled with a relative to $\mathcal{A}(\eta;a,\infty)$ and let y be an extremal which realizes this coupled point. Disconjugacy of (1.3) implies that $y(a)\ne 0$; thus by Lemma 7 and Remark 7 there exists $d>c$ such that

$$F(c) = \alpha + \gamma - \limsup_{b\to\infty}\int_a^b p, \quad F(d) > F(c).$$

Let y be a nontrivial extremal for which $y(a)=-y(d)$ and define

$$\eta = \begin{cases} y(t), & t\in[a,d] \\ y(d), & t\in[d,\infty). \end{cases}$$

Then by Lemma 2 $\mathcal{A}(\eta;a,\infty) = \lim_{b\to\infty}\eta^2(a)[\alpha+\gamma-\int_a^b p - F(d)] < 0$, a contradiction.

Finally, suppose that $\mathcal{A}(\eta;a,\infty)\ge 0$, there exists no coupled point $c\in(a,\infty)$ and the singularity condition is not satisfied, i.e. there exists $\tilde y \in W^{1,2}_{\mathrm{loc}}[a,\infty)$, $\tilde y(a) = -\tilde y(\infty)$, $\mathcal{A}(\tilde y;a,\infty)<\infty$ and

$$\liminf_{t\to\infty}\left[\tilde y^2(a)[\alpha - w_t(a)] + \tilde y^2(t)[\gamma + w_t(t)] + \frac{(\tilde y(t)-\tilde y(a))^2}{u(t)}\right] = -\varepsilon < 0. \tag{4.1}$$

Let y be an extremal such that $y(a)=\tilde y(a)$, $y(d)=\tilde y(d)$, d is sufficiently large (such solution exists because of disconjugacy of (1.3), see Lemma 1) and define

$$\eta_d(t) = \begin{cases} y(t), & t\in[a,d], \\ \tilde y(t), & t\in[d,\infty). \end{cases}$$

Then η_d is admissible and by Lemma 2

$$\begin{aligned}
\mathcal{A}(\eta_d;a,\infty) &= \alpha y^2(a) + \gamma y^2(d) + w^2 y|_a^d \\
&+ \int_a^d r^{-1}(ry'-wy)^2 + \liminf_{b\to\infty}\int_d^b (r\tilde y'^2 - p\tilde y^2) \\
&= \alpha\tilde y^2(a) + [\gamma + w(d)]\tilde y^2(d) - w(d)\tilde y^2(a) + \frac{(\tilde y(d)-\tilde y(a))^2}{u(d)} \\
&+ \liminf_{b\to\infty}\int_d^b (r\tilde y'^2 - p\tilde y^2).
\end{aligned}$$

Since $\mathcal{A}(\tilde{y}; a, b) < \infty$, we have $\liminf_{b\to\infty} \int_d^b (r\tilde{y}'^2 - p\tilde{y}^2) < \varepsilon$, if d is sufficiently large. Moreover, according to (4.1), d can be chosen such that

$$\tilde{y}^2(a)\alpha + [w(d)+\gamma]\tilde{y}^2(d) - w(a)\tilde{y}^2(a) + \frac{(\tilde{y}(d)-\tilde{y}(a))^2}{u(d)} < -\frac{2\varepsilon}{3}.$$

Consequently, we have $\mathcal{A}(\eta_d; a, \infty) < -\frac{\epsilon}{3}$, a contradiction.

Case II (b). All arguments are almost the same as in the proof of Case II (b), only instead of the function F introduced in Lemma 6 we consider the function

$$\tilde{F}(b) = \int_a^b r^{-1}(t) w_b^2(t)\, dt$$

where $w_b(t)$ is the solution of (1.6) satisfying $\int_a^b r^{-1} w = 0$. Monotonicity of this function is proved in Lemma 5.

Case II (c). The proof of this case is almost identical with the proof of Case III given by the transformation method in [2], only the solution y of (1.3) satisfying $y(a) = 0$, $y'(a) \neq 0$ is to replace by the solution satisfying $y(a) = 1$, $y'(a) = \alpha$.

Case II (d). In the proof of Case I in [3, Theorem 2], it suffices to replace the solution w of (1.6) satisfying $w(a) = \alpha$ by the solution satisfying $w(a) = \infty$.

Case III. See [2]. $\square$

Proof of Theorem 2. From the index theory of quadratic forms [4] we have the inequalities

$$c_f \leq c_p, \quad c_a \leq c_0 \tag{4.2}$$

which can be proved also directly. For example, if c_0 is the first conjugate point of a, i.e. there is a nontrivial extremal satisfying $y(a) = 0 = y(c_0)$ then it is easy to see that c_0 is also coupled with a for all types boundary conditions, so we have inequality $c_f, c_p, c_a \leq c_0$. Similarly may be proved remaining inequalities, but this reasoning gives no information concerning ordering of c_p and c_a.

Denote by w_c and w the solutions of (1.6) for which $\int_a^c r^{-1} w = 0$ and $w(a) = \alpha$, respectively. According to Lemma 7, in order to prove that the first inequality in (4.2) is strict, it suffices to show that for $c \in [a, c_f]$

$$\int_a^c r^{-1} w^2 > \int_a^c r^{-1} w_c^2$$

which is equivalent to the inequality

$$w_c(c) - w_c(a) - w(c) + \alpha > 0.$$

We have

$$w(t) = \frac{r(t)(v'(t) + \alpha u'(t))}{v(t) + \alpha u(t)}, \quad w_c(t) = \frac{r(t)[(1-v(c))u'(t) + u(c)v'(t)]}{(1-u(c))u(t) + u(c)v(t)}$$

and by a direct computation one may verify that

$$w_c(c) - w_c(a) - w(c) + \alpha = \frac{r(c)u'(c) + v(c) - 2}{u(c)} - \frac{r(c)(v'(c) + \alpha u'(c))}{v(c) + \alpha u(c)} - \alpha$$
$$= (v(c) + \alpha u(c) - 1)^2.$$

Concerning the second inequality, we have

$$\int_a^c r^{-1} w_c^2 - F(c) = \frac{4}{u(c)}\,,$$

where F is given by (3.2) with $w = w_c$, hence $c_p < c_a$.

Finally, $c_a < c_0$ provided $F(c_0-) = \infty$ and by Lemma 7 for $\bar{a} \in (a, b)$

$$F(c_0-) = F(\bar{a}) + \int_{\bar{a}}^{c_0} F'(t)\,dt$$
$$= F(\bar{a}) + \int_{\bar{a}}^{c_0} \frac{(r(t)u'(t) + 1)^2}{r(t)u^2(t)}\,dt.$$

□

Proof of Theorem 3. Lemma 7 and Remark 7 give the Riccati equation description of coupled point on the noncompact interval. Now, except for the antiperiodic boundary condition II (a), the statement follows from the fact that w given by $w(a+) = \infty$ is the only solution which exists on the whole interval (a, ∞); any other solution blows up to $-\infty$ at some finite point inside (a, ∞), see [1, Chapter I].

Concerning the antiperiodic boundary condition, if (2.4) holds then $F(\infty) = \infty$ and the statement follows using the same argument as in Lemma 7. □

Proof of Remark 2.

(i) We prove the statement only in the most difficult Case II (a); in the remaining cases the proof is much more easier. If there is no point $c \in [a, b)$ coupled with a, then by Lemma 7

$$\int_a^c r^{-1} w_c^2\,dt - \frac{4}{u(c)} \le \alpha + \gamma - \int_a^b p$$

where $\int_a^c r^{-1} w_c = 0$. Substituting for $\int_a^c r^{-1} w_c^2$ from (1.6) we have

$$\alpha + \gamma - \int_c^b p + w_c(c) - w_c(a) + \frac{4}{u(c)} \ge 0$$

for every $c \in (a, b)$ and hence

$$\liminf_{c \to b-} [\alpha + \gamma + w_c(c) - w_c(a) + \frac{4}{u(c)}] \ge 0.$$

By a direct computation we have $w_t(t) = \frac{r(t)u'(t)-1}{u(t)}$ and the expression in the singularity condition may be written in the form

$$\liminf_{t\to b-}\{y^2(a)[\alpha+\gamma+w_t(t)-w_t(a)+\frac{4}{u(t)}]+$$
$$+(\eta^2(t)-\eta^2(a))(\gamma+w_t(t))+\frac{(\eta(t)-\eta(a))^2-4y^2(a)}{u(c)}\}\geq$$
$$\geq\liminf_{t\to b-} y^2(a)[\alpha+\gamma+w_t(t)-w_t(a)+\frac{4}{u(t)}]+$$
$$+\liminf_{t\to b-}\frac{\eta(t)+\eta(a)}{u(t)}[(r(t)u'(t)-1)(\eta(t)-\eta(a))+\eta(t)-3\eta(a)\,]+$$
$$+\liminf_{t\to b-}(\eta^2(t)-\eta^2(a))\gamma.$$

The first lower limit is nonnegative, the third one is zero and if b is not conjugate with a, the second limit is also zero. If b is conjugate with a (which may happen, in view of Theorem 2, only in the antiperiodic boundary condition), the fact that $[a,c)$ does not contain a coupled point implies $r(b)u'(b)=-1$ by Lemma 6. Hence

$$\liminf_{t\to b-}\frac{\eta(t)+\eta(a)}{u(t)}[(r(t)u'(t)-1)(\eta(t)-\eta(a))+\eta(t)-3\eta(a)\,]=0,$$

i.e. the singularity condition for $\mathcal{I}(\eta;a,b)$ is satisfied.

(ii) We prove the statement, as an example, in Case II (b). Using the fact that $w_t(t)=\frac{r(t)u'(t)-1}{u(t)}$, the singularity condition in Case II (b) reads

$$\liminf_{t\to\infty}\big[\eta^2(a)(\alpha-w_t(a))+\eta^2(t)(\gamma+\frac{r(t)u'(t)-1}{u(t)})+\frac{(\eta(t)-\eta(a))^2}{u(t)}\big]\geq 0$$

and substituting $\eta(a)=0$ we get the classical singularity condition.

5. Examples

The first example shows that the nonexistence of coupled point and validity of the regularity condition need not be sufficient for nonnegativity of the singular functional.

Example 1. Consider the singular functional

$$\mathcal{F}(\eta;1,\infty)=-\eta^2(1)+\liminf_{t\to\infty}\big[\gamma\eta^2(t)+\int_1^t(\eta'^2(s)+\eta^2(s))\,ds\big]\,,\quad \gamma<-1\,,$$

over all $\eta\in W^{1,2}_{\text{loc}}\,[1,\infty)$.

The solution w of Riccati equation satisfying $w(1) = -1$ is $w(t) \equiv -1$. It holds

$$\alpha + \gamma - \int_a^\infty p = -1 + \gamma + \int_1^\infty ds = \infty,$$

i.e. the regularity condition holds and there exists no coupled point $c \in [1, \infty)$.

On the other hand, the functional is negative along the function $\eta = e^t$

$$\mathcal{F}(\eta; 1, \infty) = \lim_{b\to\infty} \big[-e^2 + \gamma e^{2b} + \int_1^b (e^{2t} + e^{2t})\, dt\big] =$$
$$-2e^2 + (\gamma + 1) \lim_{b\to\infty} e^{2b} = -\infty \qquad \text{for } \gamma < -1.$$

Remark that the singularity condition is not satisfied, because

$$\lim_{t\to\infty} e^{2t}[w(t) + \gamma] = -\infty.$$

The second example shows that the coupled point relative to $\mathcal{A}(\eta, a, b)$ may coincide with the first conjugate point.

Example 2. Consider the functional

$$\mathcal{A}(\eta; 0, \pi) = \gamma\eta^2(0) + \int_0^\pi (\eta'^2 - \eta^2)\, dt, \quad \gamma \geq 0 \tag{5.6}$$

subject to the boundary condition $\eta(0) = -\eta(\pi)$. Clearly, the first conjugate point with $t = 0$ is $t = \pi$. We will show that the interval $(0, \pi)$ does not contain a coupled point with $t = 0$. By Lemma 2, for any $c \in (0, \pi)$ and for any nontrivial extremal y on $[a, c]$ such that $y(0) = -y(c)$ and $y(t) \equiv y(c)$ for $t \in [c, \pi]$ we have

$$\begin{aligned}\mathcal{A}(y; a, \pi) &= y^2(0)[\gamma - \pi + \int_a^c \cot^2(t + \frac{\pi}{2} - \frac{c}{2}) + \frac{1}{y^2(0)} \int_a^c (y'^2 - \cot(t + \frac{\pi}{2} - \frac{c}{2})y)^2 \\ &= y^2(0)[\gamma - \pi + c + \cot(\frac{\pi}{2} + \frac{c}{2}) - \frac{4}{\sin c}].\end{aligned}$$

By Lemma 6 the function

$$F(c) = \cot(\frac{\pi}{2} - \frac{c}{2}) - \cot(\frac{\pi}{2} + \frac{c}{2}) - \frac{4}{\sin c}$$

is nondecreasing for $c \in (0, \pi)$ and

$$\lim_{c\to\pi} F(c) = \lim_{c\to\pi} \frac{4}{\sin c}(1 - \sin\frac{\pi}{2} \sin\frac{c}{2}) = 0.$$

The fact $\gamma \geq 0$ implies $F(c) \leq \gamma - \pi$ for every $c \in (0, \pi)$, i.e. there is no point coupled with 0. On the other hand, from Theorem 2 it follows that the first coupled point relative to any boundary condition of the form (1.5) has to be $\leq$ then the first conjugate point, hence, the coupled point must coincide with the conjugate point.

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