Concerning weak*-extreme points

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Abstract. Every separable nonreflexive Banach space admits an equivalent norm such that the set of the weak*-extreme points of the unit ball is discrete.

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Let K be a closed, convex and bounded subset of a Banach space X. A point x of K is called weak*-extreme if it is an extreme point of K^{**}, the weak*-closure of K in X^{**}.

James' theorem implies that the sets $\operatorname{ext} K$ and $\operatorname{ext} K^{**}$ of extreme points of, respectively, K and K^{**}, coincide if and only if the set K is weakly compact.

Godun [G] shows that the existence of an equivalent norm for which $\operatorname{ext} B$ and w^*-$\operatorname{ext} B$ do not coincide characterizes nonreflexive Banach spaces, where w^*-$\operatorname{ext} B$ denotes the set of weak*-extreme points of the unit ball B.

A theorem of Stegall [S] says that a Banach space X which fails the Radon-Nykodým property admits an equivalent norm so that the set w^*-$\operatorname{ext} B$ is empty. Moreover, the distance from X to the set $\operatorname{ext} B^{**}$ is positive.

On the other hand, Phelps [P] shows that if X has the Radon-Nykodým property then every closed, convex and bounded subset of X is the closed convex hall of its strongly exposed points (and such points are weak*-extreme).

The question arises of "how small" the set w^*-ext$B$ can be. A well-known result of Lindenstrauss and Phelps shows that in an infinite dimensional reflexive Banach space X the set ext$B$ of extreme points of the unit ball must be uncountable. In particular, if X is separable, the set ext$B$ cannot be isolated in the norm topology.

Godun, Lin, and Troyanski [GLT] show that if X is separable and nonreflexive, then it admits an equivalent norm such that w^*-$\operatorname{ext} B$ is at most countable. We observe in this note that X can even be renormed so that w^*-ext$B$ is norm-isolated. We do not know whether or not such an equivalent norm exists for any nonseparable and nonreflexive Banach space. Observe that in [LP] there is an example of a nonseparable reflexive Banach space such that ext$B$ is norm-isolated.

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In the following we denote by $\operatorname{sp} A$ (respectively, $\operatorname{co} A$) the span (respectively, the convex hull) of a set $A \subset X$.

The weak*-extreme points can be characterized as follows (see, for example, [R] and [GLT]):

Lemma 1. *Let X be a Banach space, K a closed convex bounded subset of X and $x \in K$. The following are equivalent:*

(i) *x is a weak*-extreme point of K;*
(ii) *the open slices of K, containing x, form a neighborhood base for x in the weak topology on K;*
(iii) *if $y_n,\ z_n \in K$ are such that $\lim \|x - (y_n + z_n)/2\| = 0$, than weak-$\lim(y_n - z_n) = 0$.*

We shall use the following characterization of nonreflexive Banach spaces:

Theorem 2 ([J]). *A Banach space X is nonreflexive iff for each $0 < \varepsilon < 1$ there exists a sequence $\{z_n\}$ of norm one elements so that*

$$\operatorname{dist}(\operatorname{sp}\{z_i\}_{i=1}^{n}, \operatorname{co}\{z_i\}_{i=n+1}^{\infty}) > \varepsilon$$

for any $n \in N$.

Theorem 2. *Let X be a separable nonreflexive Banach space. Then X admits an equivalent norm such that the set of weak*-extreme points of the new unit ball is isolated in the norm topology.*

Proof: Choose $\varepsilon > 0$ and a sequence $\{z_n\}$ in the unit ball of X as in Theorem 2. Clearly, $\{\pm z_n\}$ is ε-discrete. Also $\{z_n\}$ does not have a weak-cluster point in X because

$$\bigcap_{n=1}^{\infty} \overline{\operatorname{co}}\{z_i\}_{i=n}^{\infty} = \emptyset.$$

Choose some weak*-cluster point z^{**} of $\{z_n\}$ in X^{**}. Denote by Y the kernel of z^{**} in X^*. Then it is well known that for $x \in X$

$$|x| := \sup\{\langle x^*, x\rangle;\ x^* \in Y,\ \|x^*\| \le 1\}$$

defines an equivalent norm on X (cf. e.g. [GLT]). Denote the unit ball under this norm by D. Because X is separable, there is a norming sequence $\{z_k^*\}$ in the unit sphere of Y, i.e.

$$|x| = \sup\{\langle z_k^*, x\rangle;\ k \in N\} \tag{1}$$

for every $x \in X$. Since z_k^* are in the kernel of z^{**}, by passing to a subsequence of $\{z_n\}$ if necessary, we may suppose that

$$\lim_{n\to\infty} \langle z_n, z_k^* \rangle = 0 \quad \text{for} \quad k \in N. \tag{2}$$

Choose $c > 0$ such that $|x| \geq c\|x\|$ for $x \in X$ and denote $\gamma := \varepsilon c/4$. Choose a sequence $\{y_n\}$ dense in the sphere of the ball γD. Clearly the set

$$T := \{\pm(z_n \pm y_n)\}$$

is discrete and symmetric. Moreover T is bounded and

$$\gamma D \subset \overline{\mathrm{co}} T,$$

therefore $U := \overline{\mathrm{co}} T$ is a unit ball of an equivalent norm on X. Now, we have only to follow the proof in [GLT] in order to show that w^*-ext U is a subset of T.

Suppose that some weak*-extreme point x of U is not contained in T. Then by Lemma 1 there exists a sequence $\{\alpha_1 z_{n_i} + \alpha_2 y_{n_i}\}$ (where α_1 and α_2 equal 1 or -1) such that

$$\lim_{i\to\infty} \langle \alpha_1 z_{n_i} + \alpha_2 y_{n_i}, z_k^* \rangle = \langle x, z_k^* \rangle \quad \text{for} \quad k \in N. \tag{3}$$

Due to (2) we get that

$$|\langle x, z_k^* \rangle| = |\lim_{i\to\infty} \langle y_{n_i}, z_k^* \rangle| \leq \gamma, \tag{4}$$

and, because $\{z_k^*\}$ is norming, it follows that $x \in \gamma D$. Consequently, there exists some subsequence $\{y_{m_i}\}$ of the sequence $\{y_n\}$ converging in norm to x. Since

$$y_{m_i} = (y_{m_i} + z_{m_i})/2 + (y_{m_i} - z_{m_i})/2,$$

Lemma 1 implies that the sequence $\{z_{m_i}\}$ converges weakly to zero and this a contradiction to the fact that $\{z_{m_i}\}$ does not have a weak-cluster point in X.

Remark 4. *For the unit ball U constructed in the proof of the previous theorem it holds also that the distance between the sets w^*-ext U and ext $U^{**} \setminus w^*$-ext U is positive.*

Proof: If x is an extreme point of a weak*-compact set K, then the weak*-open slices containing x form a neighborhood basis for x in the weak*-topology on K. Therefore, for any $x \in \operatorname{ext} U^{**}$ there exists a sequence $\{\alpha_1 z_{n_i} + \alpha_2 y_{n_i}\}$ (where α_1 and α_2 equal 1 or -1) such that (3) and (4) are satisfied. Consequently,

$$\sup\{|\langle x, z_k^* \rangle|;\ k \in N\} \leq \gamma \quad \text{for} \quad x \in \operatorname{ext} U^{**}.$$

From the definition of T follows that

$$\sup\{\langle x, z_k^* \rangle;\ k \in N\} \geq 3\gamma \quad \text{for} \quad x \in T.$$

Since $\|z_n^*\| = 1$ for $n \in N$, it holds that the distance of the sets w^*-ext U and ext $U^{**} \setminus w^*$-ext U is greater than γ.

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