Remarks on bounded sets in $(LF)_{tv}$-spaces

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Abstract. We establish the relationship between regularity of a Hausdorff $(LB)_{tv}$-space and its properties like (K), M.c.c., sequential completeness, local completeness. We give a sufficient and necessary condition for a Hausdorff $(LB)_{tv}$-space to be an $(LS)_{tv}$-space. A factorization theorem for $(LN)_{tv}$-spaces with property (K) is also obtained.

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1. Introduction

Let $(E_n)_n$ be an increasing sequence of vector subspaces of a vector space E, whose union is E, such that every E_n is endowed with a vector topology τ_n with $\tau_{n+1}|E_n \leq \tau_n$. The space (E,τ), where τ is the finest vector topology on E such that $\tau|E_n \leq \tau_n$, $n \in \mathbb{N}$, will be called the inductive limit space and $(E_n,\tau_n)_n$ its defining sequence. We will say that (E,τ) is an

(1) $(LM)_{tv}$, if every (E_n,τ_n) is a metrizable topological vector space (tvs);
(2) $(LN)_{tv}$-space, if every (E_n,τ_n) is a locally bounded tvs;
(3) $(LF)_{tv}$-space, if every (E_n,τ_n) is an F-space, i.e. a metrizable and complete tvs;
(4) $(LB)_{tv}$-space, if every (E_n,τ_n) is a quasi-Banach space, i.e. a locally bounded and complete tvs.

It is known cf. e.g. [11, Proposition 2.2], that if every (E_n,τ_n) is a locally convex space (lcs), then τ is locally convex. In this case the corresponding inductive limit space will be called respectively (LM), (LN), (LF), (LB). Recall that a topological vector space (tvs) E is locally bounded if E has a bounded neighbourhood of zero.

Following Floret [6], [7], and Makarov [18], an inductive limit space (E,τ) with defining sequence (E_n,τ_n) will be called

(i) *regular*, if every bounded set in (E,τ) is contained in some E_m and is bounded in (E_m,τ_m);
(ii) *sequentially retractive*, if every null-sequence in (E,τ) is contained in some E_m and is a null-sequence in (E_m,τ_m).

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Following Grothendieck, a tvs E will be said to satisfy the *Mackey convergence condition* (M.c.c.) if for every null-sequence $(x_n)_n$ in E there exists a scalar sequence $(t_n)_n$, $t_n \nearrow \infty$, with $t_n x_n \to 0$. A regular $(LM)_{tv}$-space is sequentially retractive iff it satisfies M.c.c.

Grothendieck's factorization theorem [10, p. 16], implies that a Hausdorff (LF)-space is regular iff it is locally complete. Other criteria for regularity or sequential retractivity of (LF), (LB)-spaces were obtained (among others) by Floret [6], [7], [8], Neus [15], Fernandez [5], Vogt [19]. Recently we have showed [12] (extending Gilsdorf's result of [9]) that a Hausdorff $(LB)_{tv}$-space E is regular if E has property:

(K) *Every null-sequence $(x_n)_n$ in E has a subsequence $(x_{n(k)})_k$ such that the series $\sum_{k=1}^{\infty} x_{n(k)}$ converges in E.*

Note that there exist a non-sequentially complete (metrizable) tvs with property (K) ([14, Theorem 2]), and a complete tvs without property (K), cf. e.g. [12]. On the other hand every metrizable tvs with property (K) is a Baire tvs [2, 2.2].

Developing the argument used by Gilsdorf in [9] and ideas found in [6], [7], we establish the relationship between regularity of a Hausdorff $(LB)_{tv}$-space and its properties like (K), M.c.c., sequential completeness, local completeness. We give a sufficient and necessary condition for a Hausdorff $(LB)_{tv}$-space to be an $(LS)_{tv}$-space. Moreover a factorization theorem for $(LN)_{tv}$-spaces with property (K) is obtained.

We shall need the following factorization theorem , see [1, (11), pp. 57–58], and its proof.

(0) *Let F be a Baire tvs and E a Hausdorff $(LF)_{tv}$-space with defining sequence $(E_n)_n$ of F-spaces. If $T : F \to E$ is a continuous linear map, there exists $p \in \mathbb{N}$ such that $T(F) \subset E_p$ and $T : F \to E_p$ is continuous.*

By $Bd(\tau)$ we shall denote the set of all τ-bounded subsets of a tvs (E, τ); $\mathcal{F}(\tau)$ will denote the filter of all τ-neighbourhoods of zero. A sequence $(V_n)_n$ of balanced and absorbing subsets of E will be called a *string* if $V_{n+1} + V_{n+1} \subset V_n$, $n \in \mathbb{N}$; $(V_n)_n$ is *topological* if $V_n \in \mathcal{F}(\tau)$ for all $n \in \mathbb{N}$. A subset A of E will be said *pseudo-convex* if there exists a scalar $t > 0$ such that $A + A \subset tA$.

A tvs E will be called *locally complete* if for every balanced pseudo-convex bounded and closed set B in E the linear span $[B]$ endowed with the locally bounded topology generated by B is complete. It is easy to see that for lcs this definition is equivalent to the Grothendieck's one of local completeness (cf. [3, p. 152]). E will be called *locally Baire* if every bounded subset of E is contained in a bounded set B as above such that $[B]$ is a Baire tvs, cf. e.g. [9]. Every locally bounded non-complete tvs which is Baire is locally Baire but not locally complete. Every locally complete tvs with a fundamental family of pseudo-convex balanced bounded sets is locally Baire.

All tvs given in this paper are assumed to be Hausdorff.

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2. Results

We start with the following proposition; its proof is due to S. Dierolf [4].

Proposition 2.1. *Every sequentially retractive inductive limit space is regular.*

Proof: Let $(E_n, \tau_n)_n$ be a defining sequence of a tvs E under which E is sequentially retractive. Let B be a bounded subset of E; we may assume that $B \subset E_1$. Assume that B is not bounded in (E_n, τ_n), $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there exists $U_n \in \mathcal{F}(\tau_n)$ such that B is not absorbed by U_n. Thus for every $m \geq n$ there exists $b_{n,m} \in B$ with $m^{-1} b_{n,m} \notin U_n$. Consider the following sequence

$$b_{11}, 2^{-1} b_{12}, 2^{-1} b_{22}, 3^{-1} b_{13}, 3^{-1} b_{23}, 3^{-1} b_{33}, \dots .$$

This sequence converges to zero in E, hence it converges to zero in some E_n; consequently it is residually contained in U_n, a contradiction. $\square$

Lemma 2.2. *Let (E, τ) be the inductive limit space of the sequence $(E_n, \tau_n)_n$ of tvs such that*

$$(*) \qquad Bd(\tau_n) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.$$

If (E, τ) has property (K)*, then there exists for (E, τ) a defining sequence $(G_n, \gamma_n)_n$ of locally bounded Baire tvs under which (E, τ) is regular. Moreover, if every (E_n, τ_n) is locally convex, then the same is true* (*with (G_n, γ_n) normed and Baire*) *when $(*)$ is replaced by*

$$(**) \qquad Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.$$

Proof: Let $(S_n)_n$ be a sequence of balanced subsets of E such that $S_n + S_n \subset S_{n+1}$ and $S_n \in Bd(\tau_n) \cap \mathcal{F}(\tau_n)$, $n \in \mathbb{N}$. Set $A_n := \overline{S}_n^{\tau}$, $G_n := \operatorname{lin} A_n$, $n \in \mathbb{N}$. Let $K_j^n := (\alpha_n)^{-j} A_n$, $j \in \mathbb{N}$, where α_n are chosen such that $S_n + S_n \subset \alpha_n S_n$, $\alpha_n > 1$, $n \in \mathbb{N}$. Clearly $(K_j^n)_j$ forms a basis of neighbourhoods of zero for a locally bounded vector topology γ_n on G_n such that $\tau | G_n \leq \gamma_n$. Fix $n \in \mathbb{N}$. In order to prove that (G_n, γ_n) is Baire, it is enough to show that (G_n, γ_n) has property (K), cf. Introduction.

Let $(x_p)_p$ be a null-sequence in (G_n, γ_n). We may assume that $x_j \in K_j^n$, $j \in \mathbb{N}$. There exists a subsequence $(x_{p(k)})_k$ such that $\sum_{k=1}^{\infty} x_{p(k)}$ converges in τ. Since $y_m := \sum_{k=1}^{m} x_{p(k)}$, $m \in \mathbb{N}$, is γ_n-Cauchy, $y_m \in K_1^n + K_1^n \subset A_n$, $m \in \mathbb{N}$, and $(y_m)_m$ converges in $\tau | G_n$, the series $\sum_{k=1}^{\infty} x_{p(k)}$ converges in (G_n, γ_n). Consequently, (G_n, γ_n) is Baire, by [2, 2.2]. Let (E, γ) be the inductive limit space of the sequence (G_n, γ_n). Then $\tau \leq \gamma$. Let $U \in \mathcal{F}(\gamma)$ and $(U_n)_n$, be a γ-topological string with $U_1 + U_1 \subset U$. For every $m \in \mathbb{N}$ there exists $j_m \in \mathbb{N}$ such that $U_m \cap G_m \supset K^m_{j_m}$. Hence

$$U \supset U_1 + U_1 \supset \bigcup_{m=1}^{\infty} (K^1_{j_1} + K^2_{j_2} + \cdots + K^m_{j_m})$$
$$\supset \bigcup_{m=1}^{\infty} ((\alpha_1)^{-j_1})S_1 + \cdots + (\alpha_m)^{-j_m} S_m).$$

The last set belongs to $\mathcal{F}(\tau)$. [11, Proposition 2.2]; hence $\tau = \gamma$. To see that (E, τ) is regular with respect to the sequence $(G_n, \gamma_n)_n$, it is enough to show that $(A_n)_n$ is a fundamental sequence of τ-bounded sets; By [1, 16 (6)], the sequence $(A_n)_n$ is a fundamental sequence of bounded sets for the strongest vector topology ϑ on E which agrees with τ on every A_n. On the other hand $\tau = \vartheta$, cf. [13, proof of Theorem 2].

If every (E_n, τ_n) is locally convex and $(**)$ is satisfied, we choose absolutely convex sets $S_n \in Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n)$ such that $S_n + S_n \subset S_{n+1}$, $n \in \mathbb{N}$. Set $K^n_j := (2)^{-j} A_n$, $n, j \in \mathbb{N}$. To complete the proof of this case we proceed as above. □

Note that condition $(**)$ is satisfied when every (E_n, τ_n) is normed or when the inclusion map of (E_n, τ_n) into (E_{n+1}, τ_{n+1}) is compact (or precompact), $n \in \mathbb{N}$.

Corollary 2.3. *Let E be an $(LN)_{tv}$-space with property* (K) *and F an $(LF)_{tv}$-space with defining sequence $(F_n)_n$ of F-spaces. If $T : E \to F$ is a linear map with closed graph, then:*

(1) *T is continuous.*
(2) *For every bounded sets B in E there exists $m \in \mathbb{N}$ such that $T(B) \subset F_m$ and $T(B)$ is bounded in F_m.*

Proof: Combining our Lemma 2.2 with the closed graph theorem [1, (11), p. 57], one obtains the continuity of T. Now (2) follows from Lemma 2.2 and (0). □

For locally convex spaces we have even the following

Corollary 2.4. *Let (E, τ) be a lcs with property* (K). *Assume that at least one of the following conditions is satisfied.*

(a) *(E, τ) is bornological.*
(b) *(E, τ) is the inductive limit space of the sequence $(E_n, \tau_n)_n$ of lcs such that*

$$Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.$$

If F, T are defined as in Corollary 2.3, *the conclusion of Corollary* 2.3 *is also true.*

Proof: (a): Since (E, τ) is bornological with property (K), it is the inductive limit space of normed Baire spaces $[B]$, where B run over the family of absolutely convex bounded and closed subsets of E. We complete the proof as in Corollary 2.3.

(b): See the proof of Corollary 2.2. □

The following extends Theorem 5.5 of [7].

Theorem 2.5. *Let E be an $(LB)_{tv}$-space and $(E_n, \tau_n)_n$ its defining sequence consisting of quasi-Banach spaces. Consider the following conditions:*

(a) *E is sequentially retractive;*
(b) *E is sequentially complete;*
(c) *E is locally complete;*
(d) *E is locally Baire;*
(e) *E is regular;*
(f) *E has property* (K).

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) $\Leftarrow$ (f). *If E satisfies M.c.c., then all the conditions are equivalent.*

Proof: (a) $\Rightarrow$ (b): Follows from Corollary 5.3 of [7] (which also holds for $(LF)_{tv}$-spaces). (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious. (d) $\Rightarrow$ (c): This follows from the following: If B is a balanced pseudo-convex bounded and closed subset of E such that $[B]$ is Baire, then $[B]$ is continuously included in some (E_m, τ_m) (by using (0)). Since B is closed in E, it follows that $[B]$ is complete. (d) $\Rightarrow$ (e): Follows by using (0). (f) $\Rightarrow$ (e): Corollary 2.3. If E satisfies M.c.c., then (e) $\Rightarrow$ (f) $\Rightarrow$ (a) hold. □

An $(LB)_{tv}$-space (E, τ) with defining sequence (E_n, τ_n) of quasi-Banach spaces will be called an $(LS)_{tv}$-space if for every $n \in \mathbb{N}$ there exists $m > n$ such that the inclusion $(E_n, \tau_n) \to (E_m, \tau_m)$ is compact. By [17], an $(LS)_{tv}$-space is a regular B-complete (hence complete) space; hence such a space is Montel (= barrelled, see [1] for definition) for which every bounded closed set is compact and sequentially retractive.

The following extends Proposition 8.5.36 of [3].

Proposition 2.6. *Let (E, τ) be a tvs with an increasing sequence $(S_n)_n$ of balanced pseudo-convex bounded sets covering E. Then the following assertions are equivalent:*

(i) *(E, τ) is an $(LS)_{tv}$-space,*
(ii) *(E, τ) is Montel and satisfies M.c.c.*

Proof: We have only to show (ii) $\Rightarrow$ (i). Since (E, τ) is barrelled, then $(A_n)_n$, where $A_n := \overline{S_n}^{\tau}$, $n \in \mathbb{N}$, is a fundamental sequence of τ-bounded sets [1, 16 (6), (7)]. Since (by assumption) every A_n is τ-compact [1, 18 (8) and 18 (3)] apply to show that (E, τ) is a B-complete bornological DF-space. Let (E, ϑ) be the inductive limit space of quasi-Banach spaces $[A_n]$, $n \in \mathbb{N}$. Then $\tau \leq \vartheta$. Since $Bd(\tau) = Bd(\vartheta)$ and (E, τ) is bornological, then $\tau = \vartheta$, [1, 11 (3)]. For every $n \in \mathbb{N}$ there exists $m > n$ such that A_n is compact in $[A_m]$. In fact, since every A_n is τ-compact, then (by [16]) every A_n is metrizable in τ. The assumption of Grothendieck's lemma (cf. [7, p. 86]) are satisfied for $\mathcal{F}_k := \mathcal{F}(\gamma_k) \mid A_n$, $k > n$, $\mathcal{F} := \mathcal{F}(\tau)|A_n$, where γ_k is the original topology of $[A_k]$. By Grothendieck's lemma [7, p. 86], there exists $k > n$ such that $\mathcal{F}_k$ is weaker than $\mathcal{F}$; this applies to complete the proof. $\square$

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