# Universal minimal dynamical system for reals

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*Abstract.* Our aim is to give a description of $S(\mathbb{R})$ and $M(\mathbb{R})$, the phase space of universal ambit and the phase space of universal minimal dynamical system for the group of real numbers with the usual topology.

*Keywords:* ambit, Samuel compactification, minimal dynamical system

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A *dynamical system* is a triple $(G, X, \pi)$, where $G$ is a topological $T_0$ group (therefore Tychonoff), $X$ is a compact Hausdorff space and $\pi$ is a continuous action on $X$, that is, $\pi: G \times X \to X$ is a continuous map such that:

(a) $\pi(0, x) = x$ for each $x \in X$,
(b) $\pi(g + h, x) = \pi(g, \pi(h, x))$ for each $g, h \in G$ and each $x \in X$

(we use an additive notation for the group $G$, then 0 is the neutral element of $G$). If $(G, X, \pi)$ is a dynamical system then the space $X$ is called a *phase space* of the system $(G, X, \pi)$. We use the notations $\pi^g$ and $\pi_x$ for homeomorphisms $\pi^g: X \to X$ and continuous maps $\pi_x: G \to X$ defined in the following way: $\pi^g(x) := \pi(g, x)$ and $\pi_x(g) := \pi(g, x)$. The set $\pi_x(G)$ is called the *orbit* of $x \in X$ in the system $(G, X, \pi)$. If the orbit of a point $x \in X$ is dense in the phase space $X$ of dynamical system $(G, X, \pi)$ then the quadruple $(G, X, \pi; x)$ is called an *ambit* and the point $x$ a *base point* of the ambit $(G, X, \pi; x)$.

Let $(G, X, \pi)$ and $(G, Y, \varrho)$ be dynamical systems and let $\phi: X \to Y$ be a continuous map. If $\phi \circ \pi^g = \varrho^g \circ \phi$ for any $g \in G$ then $\phi$ is called a *homomorphism* of the system $(G, X, \pi)$ into the system $(G, Y, \varrho)$. If we deal with ambits $(G, X, \pi; x)$ and $(G, Y, \varrho; y)$, then homomorphism of the systems $\phi: (G, X, \pi) \to (G, Y, \varrho)$ such that $\phi(x) = y$ is called a *homomorphism of ambits.* In the case when $\phi$ is homeomorphism (surjection) of spaces then $\phi$ is called an *isomorphism* (*epimorphism*) of dynamical systems or ambits.

A dynamical system $(G, X, \pi)$ is called *minimal* if there is no proper closed non-empty set $M \subseteq X$ such that $\pi^g(M) \subseteq M$ for each $g \in G$. The system is minimal iff the orbit $\pi_x(G)$ is dense in $X$ for each $x \in X$.

An ambit $(G, X, \pi; x)$ is called *universal* for a group $G$, if for any ambit $(G, Y, \varrho; y)$ there exists an epimorphism of ambits $\phi: (G, X, \pi; x) \to (G, Y, \varrho; y)$.

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The construction of universal ambit for any group was presented by Brook in [3] (see also [6, IV.5] for other description). The phase space of universal ambit for group $G$ is Samuel compactification of $G$ with respect to its right uniformity. Equivalently, we can obtain the phase space of this ambit if we take space of all so-called regular ultrafilters with respect to a strong inclusion "$\Subset$" defined in the following way: if $F$ is closed and $U$ open in $G$ then $F \Subset U$ whenever there exists an open neighborhood $V$ of the neutral element such that $V + F \subseteq U$. A family $\mathcal{F}$ of non-empty open subsets of $G$ is called a *regular ultrafilter* whenever the following conditions hold:

(i) if $F \Subset U$, then either $U \in \mathcal{F}$ or $G \setminus F \in \mathcal{F}$,

(ii) for every $U_1, U_2 \in \mathcal{F}$ there is an open, non-empty subset $U \subseteq G$ such that $U \in \mathcal{F}$ and $\operatorname{cl} U \Subset U_1 \cap U_2$.

Let $S(G) = \{\mathcal{F} \subseteq \mathcal{P}(G) : \mathcal{F}$ is regular ultrafilter$\}$. For every open, non-empty subset $U$ of $G$, we set $\widetilde{U} = \{\mathcal{F} \in S(G) : U \in \mathcal{F}\}$. The family $\{\widetilde{U} : U$ is an open non-empty subset of $G\}$ generates compact, Hausdorff topology on $S(G)$ and the group $G$ can be embedded in $S(G)$ as a dense subspace $\{\mathcal{F}_g : g \in G\}$, where $\mathcal{F}_g = \{U : U$ is open in $G \,\&\, g \in U\}$, see [1, IV.5.], where the notion of strong inclusion corresponds with a notion of relation of subordination. Let $\pi_G : G \times S(G) \to S(G)$ be defined in the following way:

$$\pi_G(g, \mathcal{F}) := L_g(\mathcal{F}),$$

where $L_g$ is an extension on $S(G)$ of the left translation $l_g : G \to G$, expressed by the formula $L_g(\mathcal{F}) = \{g + U : U \in \mathcal{F}\}$.

**Proposition.** *The system $(G, S(G), \pi_G; 0)$ is a universal ambit for a group $G$.*

Proof: (a) The map $\pi_G$ is a continuous action.
It is not hard to see that conditions (a) and (b) of the definition of action are fulfilled. Let $L_g(\mathcal{F}) \in \widetilde{V}$, where $V$ is non-empty, open subset of $G$. There is $W \in \mathcal{F}$ such that $g + W = V$. Let $U$ be an open set such that $U \in \mathcal{F}$ and $\operatorname{cl} U \Subset W$. By the definition of "$\Subset$", there exists an open neighbourhood $H$ of the neutral element of $G$ for which $H + \operatorname{cl} U \subseteq W$. Thus $(g + H) \times \widetilde{U}$ is an open neighbourhood of $(g, \mathcal{F})$ and $\pi_G((g + H) \times \widetilde{U}) \subseteq \widetilde{V}$.

(b) The system $(G, S(G), \pi_G; 0)$ is universal.
Let $(G, X, \pi; x)$ be an ambit. The map $\pi_x : G \to X$ is uniformly continuous with respect to the ordinary inclusion on $X$ (the unique strong inclusion on compact, Hausdorff space) and strong inclusion on topological group $G$ defined above. Indeed, if $F$ is closed, $U$ is open in $X$ and $F \subseteq U$ then using compactness of $X$ we can find an open neighbourhood $H$ of 0 such that $H + \pi_x^{-1}(F) \subseteq \pi_x^{-1}(U)$, i.e. $\pi_x^{-1}(F) \Subset \pi_x^{-1}(U)$. By the theorem of Taĭmanov (see e.g. [4, 3.2.1.]) there exists a continuous map $\phi : S(G) \to X$ such that $\phi \restriction G = \pi_x$. Since $\phi(0) = x$ and $\phi \circ \pi_G^g \restriction G = \pi^g \circ \phi \restriction G$ for each $g \in G$ then $\phi$ is an epimorphism of ambits. $\square$

It is worth to notice that for discrete group $G$, $S(G)$ is equivalent to $\beta G$, Čech-Stone compactification of the discrete space $G$.

If we take a minimal closed invariant non-empty subset $M$ in the system $(G, S(G), \pi_G)$ (such a set is called shortly minimal), and consider the system $(G, M, \pi_G \upharpoonright G \times M)$, then we get a *universal minimal dynamical system* for the group $G$. This means that for any minimal dynamical system $(G, Y, \varrho)$ there is an epimorphism $\phi\colon (G, M, \pi_G \upharpoonright G \times M) \to (G, Y, \varrho)$. It is known that such universal minimal system is unique up to an isomorphism of dynamical systems (see e.g. [6, IV.3.17, IV.4.34.3]). Let $M(G)$ denote the phase space of this system.

First, we will describe space $S(\mathbb{R})$, where $\mathbb{R}$ is the additive group of real numbers with usual topology. Let $\mathbb{I}$ denote $[0;1]$, the closed interval of $\mathbb{R}$ and $\mathbb{Z}$ the group of integers.

Let $h\colon \mathbb{Z} \to \mathbb{Z}$ be the shift map, i.e. $h(n) = n+1$, and $H\colon \beta\mathbb{Z} \to \beta\mathbb{Z}$ the extension of $h$ over the Čech-Stone compactification of $\mathbb{Z}$. If we identify points $(p,1)$ with points $(H(p),0)$ in product $\beta\mathbb{Z} \times \mathbb{I}$, then we obtain a quotient space $\beta\mathbb{Z} \times \mathbb{I}/H$, which is a compactification of real line. For any integer $n \in \mathbb{Z}$ and any real number $x \in (0;1)$ we define homeomorphisms $\Lambda_n$ and $\Lambda_x$ of $\beta\mathbb{Z} \times \mathbb{I}/H$ in the following way:

$$\Lambda_n([(p,t)]_H) := [(H^n(p),t)]_H$$

and

$$\Lambda_x([(p,t)]_H) := \begin{cases} [(p,t+x)]_H & \text{if } t+x<1, \\ [(H(p),t+x-1)]_H & \text{otherwise.} \end{cases}$$

For arbitrary $x \in \mathbb{R}$, let $\Lambda_x := \Lambda_{[x]} \circ \Lambda_{\{x\}}$, where $[x]$ and $\{x\}$ denote integer and fractional part of $x$ respectively. Define a map $\varrho\colon \mathbb{R} \times (\beta\mathbb{Z} \times \mathbb{I}/H) \to \beta\mathbb{Z} \times \mathbb{I}/H$ by the formula

$$\varrho(x,w) := \Lambda_x(w).$$

Let $q$ denote the quotient map from $\beta\mathbb{Z} \times \mathbb{I}$ onto $\beta\mathbb{Z} \times \mathbb{I}/H$.

**Lemma.** *The system $(\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho; z)$, where $z = [(0,0)]_H$ is an ambit.*

Proof: Conditions (a) and (b) of the definition of action are obviously fulfilled. We will show continuity of $\varrho$ at points of the form $(n,w)$, where $n \in \mathbb{Z}$. Let $V$ be an open neighbourhood of $\varrho(n,w)$. Suppose, that $w = [(p,t)]_H$ and $t \in (0;1)$. Since $\varrho(n,w) = [(H^n(p),t)]_H$ then there exist an open set $W \subseteq \beta\mathbb{Z}$ and $\varepsilon > 0$ such that $(H^n(p),t) \in W \times (t-\varepsilon; t+\varepsilon) \subseteq q^{-1}(V)$. A set

$$(n-\frac{\varepsilon}{2}; n+\frac{\varepsilon}{2}) \times q\big(H^{-n}(W) \times (t-\frac{\varepsilon}{2}; t+\frac{\varepsilon}{2})\big)$$

is an open neighbourhood of $(n,w)$ and its image by $\varrho$ is contained in $V$. If $w = [(p,1)]_H$ then we can find an open set $W \subseteq \beta\mathbb{Z}$ and $\varepsilon > 0$ such that $(H^n(p),1) \in W \times (1-\varepsilon;1] \subseteq q^{-1}(V)$ and $(H^{n+1}(p),0) \in H(W) \times [0;\varepsilon) \subseteq q^{-1}(V)$. In this situation a set

$$U = (n-\frac{\varepsilon}{2}; n+\frac{\varepsilon}{2}) \times q\big((H^{-n}(W) \times (1-\frac{\varepsilon}{2};1]) \cup (H^{-n+1}(W) \times [0;\frac{\varepsilon}{2}))\big)$$

is open neighbourhood of $(n, w)$ and $\varrho(U) \subseteq V$. Proof of continuity of $\varrho$ at points of the form $(x, w)$, where $x \in (0;1)$ is similar. Let $x \in \mathbb{R} \setminus \mathbb{Z}$ be arbitrary. If $V$ is an open set such that $\varrho(x, w) \in V$ then there exist open sets $V_1$, $V_2$ such that $\{x\} \in V_1 \subseteq (0;1)$, $w \in V_2 \subseteq \beta\mathbb{Z} \times \mathbb{I}/H$ and $\varrho(V_1 \times V_2) \subseteq \Lambda^{-1}_{[x]}(V)$. Obviously $\varrho(([x] + V_1) \times V_2) \subseteq V$.

The orbit of point $z = [(0,0)]_H$ equals $\mathbb{Z} \times \mathbb{I}/H$, thus is dense in $\beta\mathbb{Z} \times \mathbb{I}/H$. □

**Theorem.** *The universal ambit for the group of reals with usual topology is isomorphic to the ambit* $(\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho; z)$.

Proof: Since $(\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho; z)$ is an ambit, we have an epimorphism

$$\phi\colon (\mathbb{R}, S(\mathbb{R}), \pi_{\mathbb{R}}; 0) \to (\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho; z).$$

Since $\phi(0) = z$, $\phi \restriction \mathbb{R}\colon \mathbb{R} \to \mathbb{Z} \times \mathbb{I}/H$ is a map of form $\phi(x) = [([x], \{x\})]_H$. We will show that $\phi$ is one-to-one. Let $\mathcal{F}, \mathcal{F}' \in S(\mathbb{R})$ and $\mathcal{F} \neq \mathcal{F}'$. Let $U \in \mathcal{F}$, $U' \in \mathcal{F}'$ and $U \cap U' = \emptyset$. We can find open sets $V, V'$ such that $V \in \mathcal{F}, V' \in \mathcal{F}'$ and $\mathrm{cl}_{\mathbb{R}} V \Subset U, \mathrm{cl}_{\mathbb{R}} V' \Subset U'$. By the definition of "$\Subset$", there is $\varepsilon > 0$ such that $((-\varepsilon;\varepsilon) + \mathrm{cl}_{\mathbb{R}} V) \cap \mathrm{cl}_{\mathbb{R}} V' = \emptyset$. Obviously, $\mathcal{F} \in \mathrm{cl}_{S(\mathbb{R})} \mathrm{cl}_{\mathbb{R}} V$ and $\mathcal{F}' \in \mathrm{cl}_{S(\mathbb{R})} \mathrm{cl}_{\mathbb{R}} V'$. Let denote $F_1 = \phi(\mathrm{cl}_{\mathbb{R}} V)$ and $F_2 = \phi(\mathrm{cl}_{\mathbb{R}} V')$. In additon, for brevity sake, we denote $\beta\mathbb{Z} \times \mathbb{I}/H$ by $K$ and $\beta\mathbb{Z} \times \mathbb{I}$ by $K'$. It suffices to prove that $\mathrm{cl}_K F_1 \cap \mathrm{cl}_K F_2 = \emptyset$. Suppose there exists $[(p,t)]_H \in \mathrm{cl}_K F_1 \cap \mathrm{cl}_K F_2$. Let $\delta < \varepsilon/2$.

*Case* 1. $0 < t < 1$

In this case $(p,t) \in \mathrm{cl}_{K'} q^{-1}(F_1) \cap \mathrm{cl}_{K'} q^{-1}(F_2)$. Let

$$A_j = \{k \in \mathbb{Z} : (\{k\} \times (t-\delta; t+\delta)) \cap q^{-1}(F_j) \neq \emptyset\}, \quad j \in \{1,2\}.$$

One can verify that $A_1$,$A_2 \in p$, so $A_1 \cap A_2 \in p$. Thus there exists $k \in A_1 \cap A_2$. By the definition of $A_1$ and $A_2$ there are $[(k,t_1)]_H \in F_1$ and $[(k,t_2)]_H \in F_2$ where $|t_1 - t_2| < 2\delta < \varepsilon$. This is impossible because $((-\varepsilon;\varepsilon) + \mathrm{cl}_{\mathbb{R}} V) \cap \mathrm{cl}_{\mathbb{R}} V' = \emptyset$.

*Case* 2. $t \in \{0,1\}$

We can assume that $t = 1$, because for $t = 0$ the proof is analogous. Then $q^{-1}([(p,1)]_H) = \{(p,1), (H(p),0)\}$ and for $j \in \{1,2\}$ we have that $(p,1) \in \mathrm{cl}_{K'} q^{-1}(F_j)$ or $(H(p),0) \in \mathrm{cl}_{K'} q^{-1}(F_j)$. Let us consider the case $(p,1) \in \mathrm{cl}_{K'} q^{-1}(F_1)$ and $(H(p),0) \in \mathrm{cl}_{K'} q^{-1}(F_2)$ (we can proceed quite similarly as with other cases). A set

$$A_1 = \{k \in \mathbb{Z} : (\{k\} \times (1-\delta;1]) \cap q^{-1}(F_1) \neq \emptyset\}$$

belongs to $p$, and by similar reasons a set

$$A_2 = \{k \in \mathbb{Z} : (\{k\} \times [0;\delta)) \cap q^{-1}(F_2) \neq \emptyset\}$$

belongs to $H(p)$. Since $A_2 \in H(p)$ then $A_2 - 1 \in p$. Let $k \in A_1 \cap (A_2 - 1)$. Thus there exist points $[(k,t_1)]_H \in F_1$, $[(k+1,t_2)]_H \in F_2$ such that $1 - \delta < t_1 \leqslant 1$ and $0 \leqslant t_2 < \delta$; a contradiction.

So, $\phi$ is the isomorphism of ambits. □

**Corollary.** *The phase space of the universal minimal dynamical system for the group* $\mathbb{R}$ *is homeomorphic to the quotient space* $E(D^{2^\omega}) \times \mathbb{I}/H$*, where* $E(D^{2^\omega})$ *denote the absolute of the Cantor cube* $D^{2^\omega}$ *and* $H$ *is a homeomorphism of* $E(D^{2^\omega})$.

Proof: As the systems $(\mathbb{R}, S(\mathbb{R}), \pi_{\mathbb{R}})$ and $(\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho)$ are isomorphic, the minimal subsets of these systems are isomorphic. In order to describe the structure of $M(\mathbb{R})$, it suffices to consider arbitrary minimal subset in the system $(\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho)$. Let $M$ be a minimal non-empty closed and invariant subset in $\beta\mathbb{Z}$ for $H\colon \beta\mathbb{Z} \to \beta\mathbb{Z}$. Then $M$ is homeomorphic to $M(\mathbb{Z})$, the phase space of universal minimal dynamical system for group $\mathbb{Z}$. It is not hard to see that a set $M \times \mathbb{I}/H \subseteq \beta\mathbb{Z} \times \mathbb{I}/H$ is closed and invariant in the system $(\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho)$. Moreover, an orbit of any point of $M \times \mathbb{I}/H$ is dense in $M \times \mathbb{I}/H$. So, $M \times \mathbb{I}/H$ is a minimal subset in $(\mathbb{R}, \beta\mathbb{Z} \times \mathbb{I}/H, \varrho)$. Balcar and Błaszczyk proved in [2] that the space $M(\mathbb{Z})$ is an absolute of the Cantor cube $D^{2^\omega}$. Therefore, we can obtain $M(\mathbb{R})$ if in the product of the absolute of Cantor cube $D^{2^\omega}$ and closed segment $[0;1]$; the points $(x, 1)$ and $(H(x), 0)$ are identified. □

**Remark.** Since homeomorphism $H \restriction M$ has dense orbit then the space $M(\mathbb{R}) \overset{\text{top}}{=} M \times \mathbb{I}/H$ is an indecomposable continuum (see [5]). Therefore, $M(\mathbb{R})$ is so-called generalized solenoid.

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