A remark on a paper by Bhattacharya and Leonetti

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Abstract. We prove higher integrability for the gradient of bounded minimizers of some variational integrals with anisotropic growth.

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Introduction

In this note we refer to Bhattacharya and Leonetti's paper [1]; in the sequel formulas containing two numbers and a dot in between, like (1.2), are taken from [1]; on the other hand, formulas containing only one number, like (3), are new and appear only in the present note. For motivation, definitions and further references we address the reader to [1]. We study regularity for functions $u : \Omega \to \mathbb{R}^N$ minimizing the variational integral

$$I(u) = \int_\Omega F(Du(x))\,dx, \tag{1.1}$$

where $F(\xi)$ behaves like the model example

$$\frac{1}{2}\sum_{i=1}^{n-1} |\xi_i|^2 + \frac{1}{p}(1 + |\xi_n|^2)^{p/2},$$

precise conditions are given by (1.2), … , (1.6). The aim of this note is to show that the additional assumption "u is bounded" allows us to improve the result contained in [1] in dimension 4; also, it simplifies the proof very much. In the scalar case $N = 1$, Moscariello-Nania [4] and Fusco-Sbordone [2], [3], proved that minimizers are locally bounded.
More precisely, we have the following

Theorem. *Let $u : \Omega \to \mathbb{R}^N$ verify*

$$u \in W^{1,1}(\Omega), \quad D_i u \in L^2(\Omega), \quad i = 1, \dots, n-1, \quad D_n u \in L^p(\Omega), \tag{1}$$

Ω bounded, open $\subset \mathbb{R}^n$, $n \geq 2$, where

$$1 < p < 2 \qquad \text{if} \quad n = 2, 3, 4, \tag{2}$$

$$(1.10) \qquad 2-4/n<p<2 \qquad \textit{if} \quad n\geq 5.$$

Assume that

$$(3) \qquad u\in L^{\infty}(\Omega),$$

u minimizes the variational integral (1.1) *and* (1.2), ..., (1.5) *are fulfilled, then*

$$(1.11) \qquad D_n u\in L^2_{\mathrm{loc}}(\Omega).$$

Furthermore, the second weak derivatives exist:

$$(4) \qquad D_i Du\in L^2_{\mathrm{loc}}(\Omega), \quad i=1,\dots,n-1 \qquad \textit{and} \qquad D_n Du\in L^p_{\mathrm{loc}}(\Omega).$$

This theorem and [2], [3], yield the following

Corollary. *In the scalar case, that is, when* $u:\Omega\to\mathbb{R}$, *we assume* (1), (2), (1.10). *If u minimizes the variational integral* (1.1), *if* (1.2), ..., (1.5) *are fulfilled and* (0.2) *holds with* $q_1=\dots=q_{n-1}=2$, $q_n=p$, *then u is locally bounded in* Ω *and* (1.11), (4), *hold true.*

Proof of the Theorem: We argue as in [1] and we arrive at (3.8); in the sequel, C_i will denote a positive constant, independent of h. Since we only know that $D_n u\in L^p$, the integral corresponding to $s=n$ in (3.8) is dealt with as follows. Let us assume that

$$(5) \qquad D_n u\in L^{\sigma}_{\mathrm{loc}}(\Omega),$$

for some σ verifying $p\leq\sigma<2$. We write

$$\int_{B_R}|\tau_{n,h}u|^2\,dx=\int_{B_R}|\tau_{n,h}u|^{\sigma}|\tau_{n,h}u|^{2-\sigma}\,dx.$$

We recall our assumption (3): u is bounded; then $|u(y)|\leq C_6$ for every $y\in B_{2R}$, thus $|\tau_{n,h}u(x)|^{(2-\sigma)}\leq(2C_6)^{(2-\sigma)}$ for every $x\in B_R$ and every $h:|h|<R$. Since we assumed (5), we may apply Lemma 2.1 with $t=\sigma$ and we get

$$(6) \qquad \int_{B_R}|\tau_{n,h}u|^2\,dx\leq C_7|h|^{\sigma}\int_{B_{2R}}|D_n u|^{\sigma}=C_8|h|^{\sigma}.$$

Since $\sigma<2$ and $R\leq 1$, (3.8), (6) and (3.7) yield

$$\sum_{s=1}^{n}\int_{B_\rho}|\tau_{s,h}\hat{V}(Du)|^2\,dx\leq C_9|h|^{\sigma} \quad \forall\, h:|h|<R.$$

Now via Lemma 2.3 we improve the integrability:

$$\hat{V}(Du) \in L^r_{\mathrm{loc}}(\Omega) \quad \forall\, r < 2n/(n-\sigma).$$

If we recall (3.5), then

$$D_n u \in L^t_{\mathrm{loc}}(\Omega) \qquad \forall\, t < pn/(n-\sigma) = \hat{t}(\sigma). \tag{7}$$

So we started from (5) and we boosted the integrability up to (7); let us estimate $\hat{t}(\sigma) - \sigma$:

$$\hat{t}(\sigma) - \sigma = \frac{\sigma^2 - n\sigma + pn}{n - \sigma} = \frac{f(\sigma)}{g(\sigma)}.$$

When $p \le \sigma < 2$, $0 < g(\sigma) \le n - p$. The function f is decreasing in $(-\infty, n/2)$ and increasing in $(n/2, +\infty)$, thus it achieves its minimum value for $\sigma = n/2$: $f(\sigma) \ge f(n/2) = n(4p-n)/4$; such a value turns out to be positive when $n = 2$ or $n = 3$ or $n = 4$. When $5 \le n$, we have $2 < n/2$, thus $f(\sigma)$ decreases for $\sigma \in [p,2]$, so that

$$f(\sigma) \ge f(2) = 4 - 2n + pn = n(p - (2 - 4/n)) > 0,$$

since we assumed (1.10). We can summarize as follows: because of (2) and (1.10),

$$\hat{t}(\sigma) - \sigma \ge \frac{\min_{\sigma \in [p,2]} f(\sigma)}{n - p} = \delta(n,p) > 0,$$

for every $\sigma \in [p, 2)$. Let us recall (5) and (7): we have proved that, if for some $\sigma \in [p,2)$ we have $D_n u \in L^{\sigma}_{\mathrm{loc}}$, then we also have $D_n u \in L^{\sigma+\delta/2}_{\mathrm{loc}}$. This allows us to start a bootstrap argument which completes the proof of (1.11). The higher differentiability (4) follows from (1.11) as it is shown in [1]. $\square$

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References

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