Almost split sequences and module categories : A complementary view to Auslander-Reiten Theory

Ariel Fernández

Abstract. We take a complementary view to the Auslander-Reiten trend of thought: Instead of specializing a module category to the level where the existence of an almost split sequence is inferred, we explore the structural consequences that result if we assume the existence of a single almost split sequence under the most general conditions. We characterize the structure of the bimodule ${}_{\Delta}\mathrm{E}xt_R(C,A)_{\Gamma}$ with an underlying ring R solely assuming that there exists an almost split sequence of left R-modules $0 \to A \to B \to C \to 0$. Δ and Γ are quotient rings of $\mathrm{E}nd({}_RC)$ and $\mathrm{E}nd({}_RA)$ respectively. The results are dualized under mild assumptions warranting that ${}_{\Delta}\mathrm{E}xt_R(C,A)_{\Gamma}$ represent a Morita duality. To conclude, a reciprocal result is obtained: Conditions are imposed on ${}_{\Delta}\mathrm{E}xt_R(C,A)_{\Gamma}$ that warrant the existence of an almost split sequence.

Keywords: almost split sequence, Morita duality

Classification: 16G70

1. Preliminaries and notation

This work is motivated by the need to investigate structural properties of ${}_{\Delta}\mathrm{Ext}\,_R(C,A)_{\Gamma}$ as a $\Delta - \Gamma$ bimodule under the assumption that there exists an almost split sequence $0 \to A \to B \to C \to 0$ (also denoted (a,b) where $a : A \to B$ and $b : B \to C$) of left modules over a ring R. Δ and Γ are quotient rings of $\mathrm{End}({}_RC)$ and $\mathrm{End}({}_RA)$ respectively ([1]). Thus, instead of finding conditions under which the existence of an almost split sequence is warranted ([1], [2]), we take the complementary perspective: We assume from the start the existence of an almost split sequence under the most general conditions and infer structural properties of the underlying ring R. On the other hand, the Auslander-Reiten philosophy shared by Zimmermann [2] has always been to specialize the module categories over the rings R, Δ and Γ, so that an almost split sequence may be constructed ([1]) or may be shown to exist ([2]).

Throughout the paper we adopt standard notation. Thus $A, B, C, X, Y, Z, \dots$ denote left R-modules over the ring R. Moreover, following [1], [2], we denote:

$P(X,Y) = \{f \in \mathrm{Hom}\,_R(X,Y) \mid f$ factors over a projective R-module$\}$

$I(X,Y) = \{f \in \mathrm{Hom}\,_R(X,Y) \mid f$ factors over an injective R-module$\}$

Financial support was provided by Fundación Antorchas (Argentina) and the J.S. Guggenheim Memorial Foundation (USA)

$\mathrm{Hom}_R(X,Y) = \mathrm{Hom}_R(X,Y)/P(X,Y)$; $\overline{\mathrm{Hom}}_R(X,Y) = \mathrm{Hom}_R(X,Y)/I(X,Y)$
$D = \mathrm{End}({}_RC)$; $G = \mathrm{End}({}_RA)$; $\Delta = \underline{\mathrm{End}}({}_RC)$; $\Gamma = \overline{\mathrm{End}}({}_RA)$.

Since R need not be an Artin algebra ([1], [2]), we provide a general definition of almost split sequence: A nonsplitting short exact sequence denoted $0 \to A \to B \to C \to 0$ with homomorphisms $a : A \to B$ and $b : B \to C$ of left modules over any ring R is called almost split if for every $g \in \mathrm{Hom}_R(A,X)$, with g any homomorphism which is not a splitting monomorphism, there exists $g' \in \mathrm{Hom}_R(B,X)$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "g"'] & & B \arrow[dl, "g'"] \\ & X & \end{tikzcd} \tag{1}$$

and the dual statement is true for every $d \in \mathrm{Hom}_R(Y,C)$ which is not a splitting epimorphism.

2. Results

The following theorem generalizes a result proven for Artin algebras by Auslander and Reiten ([1, Theorem 3.3]). By contrast with theirs, our proof is **elementary** since it does not make use of functor categories:

Theorem 1. *For every module ${}_RX$, the map $\overline{\mathrm{Hom}}_R(X,A) \ni \overline{g} \to \mathrm{Ext}(C,g) \in \mathrm{Hom}_\Delta(\mathrm{Ext}_R(C,X), \mathrm{Ext}_R(C,A))$ is a monomorphism of right Γ-modules.*

Proof: Obviously the map is a homomorphism of right Γ-modules. Let $\mathrm{Ext}(C,g)$ be zero. We have to show that g factors over an injective module. This is shown in [3]. $\square$

The map dealt with in this theorem is actually an isomorphism under the relatively mild additional assumptions R semiperfect and ${}_RC$ finitely presented (cf. [3]). Remarkably, no conditions need to be imposed upon ${}_RX$, in contrast with the results of Auslander and Reiten [1] for Artin algebras.

3. Dualization of the results

Let us fix the setting of reference [3]: R semiperfect; ${}_RC$ finitely presented and $\mathrm{End}({}_RC)$ local ring. Let us introduce further notation: TrC_R = transpose of ${}_RC$; $T = \mathrm{End}(TrC_R)$; ${}_TE$ = injective hull of $T/Ra(T)$. All the notation is standard (cf. [1]). There are a number of instances in which ${}_TE_G$ defines a Morita duality ([3]):

(a) ${}_RA$ is finitely presented and TrC_R is purely injective (each pure exact sequence $0 \to TrC_R \to M_R$ splits).

(b) T is a left Artin ring and ${}_TE$ is finitely generated.
(c) TrC_R is simple.
(d) R is an Artin algebra.
(e) R is a ring of finite module type.

We have already shown ([3]) that if ${}_TE_G$ defines a Morita duality, then ${}_\Delta\mathrm{Ext}_R(C,A)_\Gamma$ is the induced Morita duality. This result is paramount to introduce a dualization of the context presented in the previous section. Accordingly, we shall prove the following results:

Proposition 1. *Let ${}_TE_G$ be a Morita duality. For $n \in N$, let ${}_RY$ be a direct summand of ${}_RC^n$, or let ${}_RC$ be self-projective and ${}_RY$ be an epimorphic image of ${}_RC^n$. Then ${}_T\mathrm{Hom}_R(C,Y)$ is reflexive with respect to ${}_TE_G$, and ${}_\Delta\underline{\mathrm{Hom}}_R(C,Y)$ is reflexive with respect ${}_\Delta\mathrm{Ext}_R(C,A)_\Gamma$.*

PROOF: Under the given assumptions, there is an epimorphism ${}_TT^n \approx {}_T\underline{\mathrm{Hom}}_R(C,C^n) \to {}_T\underline{\mathrm{Hom}}_R(C,Y) \to 0$ which yields the first statement. The second statement follows from well-known properties of the induced Morita duality which one obtains from ${}_TE_G$ by passing over from D and G to Δ and Γ.

At this point, we shall prove the following

Theorem 2. *Let ${}_TE_G$ be a Morita duality, then the following statements are equivalent:*

(1) ${}_T\underline{\mathrm{Hom}}_R(C,Y)$ *is reflexive with respect to* ${}_TE_G$.
(2) $\underline{\mathrm{Hom}}_R(C,Y) \ni d \to \mathrm{Ext}(d,A) \in \mathrm{Hom}_\Gamma(\mathrm{Ext}_R(Y,A),\mathrm{Ext}_R(C,A))$ *is an* **isomorphism** *of left T-modules. In this case,* ${}_\Delta\underline{\mathrm{Hom}}_R(C,Y)$ *and* $\mathrm{Ext}_R(Y,A)_\Gamma$ *are reflexive with respect to* ${}_\Delta\mathrm{Ext}_R(C,A)_\Gamma$.

PROOF: Let Ω denote the composition of the G-isomorphisms $\mathrm{Ext}_R(Y,A) = \mathrm{Ext}_R(Y,\mathrm{Hom}_T(TrC,E)) \approx \mathrm{Hom}_T(Tor^R(TrC,Y),E) \approx \mathrm{Hom}_T(\underline{\mathrm{Hom}}_R(C,Y),E)$. Then the following diagram commutes, where Σ is the evaluation map:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_R(C,Y) & \xrightarrow{\mathrm{Ext}(-,A)} & \mathrm{Hom}_G(\mathrm{Ext}_R(Y,A),\mathrm{Ext}_R(C,A)) \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\approx} \\ \mathrm{Hom}_G(\mathrm{Hom}_T(\underline{\mathrm{Hom}}_R(C,Y),E),E) & \xrightarrow[\mathrm{Hom}(\Omega,E)]{} & \mathrm{Hom}_G(\mathrm{Ext}_R(Y,A),E) \end{array} \tag{2}$$

and thus our assertion follows. □

Proposition 2. *Let ${}_RA$ be finitely presented and TrC_R be a purely injective module. Let ${}_RX$ be any finitely presented module. Then, the following statements hold:*

(1) $\mathrm{Hom}_R(X,A)_G$ *and* ${}_T\mathrm{Ext}_R(C,X)$ *are reflexive with respect to* ${}_TE_G$.

(2) $\overline{\mathrm{Hom}}_R(X,A)_G \approx \mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$ *and* $_T TrC \otimes {}_R X \approx {}_T\mathrm{Hom}_G(\mathrm{Hom}_R(X,A),E)$.

(3) $\overline{\mathrm{Hom}}_R(X,A)_\Gamma$ *and* $_\Delta\mathrm{Ext}_R(C,X)$ *are reflexive with respect to* $_\Delta\mathrm{Ext}_R(C,A)_\Gamma$.

Proof: Under the above assumptions $_T E_G$ is a Morita duality ([3]), the module TrC_R is reflexive with respect to $_T E_G$ and there exists an isomorphism $\mathrm{Hom}_G(A,E)_R \approx TrC_R$.
Let $w : {}_T\mathrm{Hom}_G(A,E) \otimes {}_R X \to {}_T\mathrm{Hom}_G(\mathrm{Hom}_R(X,A),E)$ denote the natural isomorphism, and let Σ and Σ' be the evaluation maps from $_T TrC$ and $\mathrm{Hom}_R(X,A)_G$ into their biduals with respect to $_T E_G$. Then (1) follows from the commutativity of the diagram:
(3)

$$\begin{array}{ccc}
\mathrm{Hom}_R(X,\mathrm{Hom}_T(TrC,E))=\mathrm{Hom}_R(X,A) & \xrightarrow[\Sigma]{} & \mathrm{Hom}_T(\mathrm{Hom}_G(\mathrm{Hom}_R(Y,A),E),E) \\
\downarrow \mathrm{adj} & & \downarrow \mathrm{Hom}(\Omega,E) \\
\mathrm{Hom}_T(TrC\otimes_R X,E) & \xleftarrow[\mathrm{Hom}(\Sigma'\otimes X,E)]{} & \mathrm{Hom}_T(\mathrm{Hom}_G(A,E)\otimes_R X,E)
\end{array}$$

As we have the epimorphism $\mathrm{Hom}_R(X,A)_G \to \mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$, the module $\mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$ is reflexive and, consequently, $_T(\mathrm{Ext}_R(C,X))$ is also reflexive.

(2) We have the isomorphisms:
$\overline{\mathrm{Hom}}_R(X,A)_G \approx \mathrm{Hom}_T(\mathrm{Ext}_R(C,X),\mathrm{Ext}_R(C,A))_G \approx$
$\mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$.
The second statement follows from
$_T\mathrm{Hom}_G(\mathrm{Hom}_R(X,A),E) \approx {}_T\mathrm{Hom}_G(A,E) \otimes {}_R X \approx {}_T TrC \otimes {}_R X$.
(3) is a consequence of (1). □

4. Under what conditions do we find an almost split sequence?

At this point we are in a position to prove a plausible reciprocal of the results expounded previously. The conditions under which the existence of an almost split sequence is warranted are less demanding than those by Zimmermann ([2]), since the left modules are not required to be finitely presented.

Theorem 3. *Assume the following conditions are satisfied (standard notation is followed):* $_{\Delta'}\mathrm{Ext}_R(C',A')$ *is injective;* $\mathrm{Soc}({}_{\Delta'}\mathrm{Ext}_R(C',A'))$ *is simple and essential in* $_{\Delta'}\mathrm{Ext}_R(C',A')$*;* $\mathrm{Soc}(\mathrm{Ext}_R(C',A')_{\Gamma'}) \supseteq \mathrm{Soc}({}_{\Delta'}\mathrm{Ext}_R(C',A'))$*;* D' *and* G' *are local rings, and for every* $_R X$*, the map* $\mathrm{Ext}(C,-) : \mathrm{Hom}_R(X,A') \ni g \to \mathrm{Ext}(C',g) \in \mathrm{Hom}_{\Delta'}(\mathrm{Ext}_R(C',X),\mathrm{Ext}_R(C',A'))$ *is surjective. Then every nonzero element* $(a',b') \in \mathrm{Soc}({}_{\Delta'}\mathrm{Ext}_R(C',A'))$ *is almost split.*

Proof: Let $g \in \mathrm{Hom}_R(A',X)$ be a homomorphism which has no factorization over a'. As $\mathrm{Ext}(C',g)$ operates nonzero on a simple essential submodule of

$\operatorname{Ext}_R(C', A')$, it is a monomorphism. From the injectivity of ${}_{\Delta'}\operatorname{Ext}_R(C', A')$, it follows that $\operatorname{Ext}(C', g)$ splits, and, from the assumption that $\operatorname{Ext}(C', -)$ is an epimorphism, we obtain $g' \in \operatorname{Hom}_R(X, A')$ such that $\operatorname{Ext}(C', gg') = \mathrm{id}$. Thus, the composition gg' is an isomorphism, since otherwise it would follow that $gg' \in Ra(G')$ and $gg'(a', b') = 0$, which is a contradiction. Hence g is a splitting monomorphism and we have shown that (a', b') is almost split on the left side. The lifting property on the right side of the sequence also holds since we have assumed that D' is a local ring. $\square$

Acknowledgements. Early versions of this work were carried out at Yale under the guidance of Prof. Nathan Jacobson, whose encouragement is appreciated. My heartfelt thanks go to Profs. María Inés Platzeck (UNS, Argentina) and Boris Kunin (Yale) for their invaluable help.

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Departamento e Instituto de Matemática, Universidad Nacional del Sur-Consejo Nacional de Investigaciones Científicas y Técnicas, Avenida Alem 1253, Bahía Blanca 8000, Argentina

(*Received* January 10, 1995, *revised* April 12, 1995)