On resolvable spaces and groups

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Abstract. It is proved that every uncountable ω-bounded group and every homogeneous space containing a convergent sequence are resolvable. We find some conditions for a topological group topology to be irresolvable and maximal.

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1. Introduction

Hewitt [H] introduced and studied the properties of being irresolvable and/or maximal for topological spaces. He constructed the first examples of irresolvable spaces by generating maximal topologies. Several years later Malykhin [M], assuming some extra set-theoretic axioms, gave the first and, as far as we know, the only example of a countable irresolvable topological group, which is maximal as well. This raises the question about the existence of non-maximal irresolvable groups. In the case of general topological spaces there are several examples of such spaces ([A], [P]). This paper contributes to the problem of irresolvable non-maximal groups by establishing conditions on the weight of maximal topological groups. The results about irresolvable groups hold under the assumption of the combinatorial principle $p = \mathfrak{c}$. We also prove the resolvability of certain uncountable α-bounded groups and of a homogeneous space containing convergent sequences.

The notation and definitions used in this paper is the same as in the related literature, but the following are specified: Ω represents the group of finite subset of ω with the symmetric difference as a group operation. The symbol $\langle H\rangle$ denotes the subgroup generated by the set H. An ordinal is the set of its predecessors. We only consider Hausdorff non-discrete topological groups and Hausdorff spaces which are dense in themselves. The symbol e denotes the identity element of a group. For the weight and density of a topological space X we use $w(X)$ and $d(X)$. The symbol ω is the first infinite ordinal and $\mathfrak{c}$ is the least ordinal of the same cardinality as $\{x : x \subseteq \omega\}$. The closure of a subset U of X is denoted by U^-.

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2. Definitions and preliminary results

This section contains the relevant concepts, definitions and results in the theory of maximal and irresolvable spaces which form the basis of our work.

The following two definitions were introduced by E. Hewitt.

Definition 2.1. A space X is k-resolvable ($2 \leq k$) if X contains k disjoint dense subsets. If $k = 2$, we simply say that X is resolvable. A space which is not resolvable is called irresolvable.

Definition 2.2. A topological space (X, τ) dense in itself is *maximal* (and τ is a *maximal topology*) if every dense in itself topology τ' on X with $\tau \subseteq \tau'$ coincides with τ.

Definition 2.3 [G]**.** Let α be an infinite cardinal. A topological group X is called α-bounded if for every neighborhood U of the identity there exists a subset $K \subseteq X$ with $|K| \leq \alpha$ such that $X = K \cdot U$.

It is not hard to see that every subgroup of an α-bounded topological group is α-bounded. The metrizable $\aleph_0$-bounded groups are separable, and every $\aleph_0$-bounded group may be embedded as a subgroup of a product of separable groups ([G]).

We will use the following results.

Theorem 2.4 [CF]**.** *Let $X = \bigcup_{i \in I} X_i$ be a space with each X_i resolvable. Then X is resolvable. In particular, every homogeneous spaces containing a resolvable subspace is itself resolvable.*

Definition 2.5. *A collection of sets S is called centered if the intersection of every finite subfamily of S is infinite.*

We will use the following combinatorial principle, which is a consequence of the Martin's Axiom.

$p = \mathfrak{c}$**.** *For each centered collection $\mathcal{A}$ of fewer than $\mathfrak{c}$ subsets of ω, there is an infinite $B \subseteq \omega$ such that $B \setminus A$ is finite for all $A \in \mathcal{A}$.*

3. Results

Theorem 3.1. *Let G be a topological group, $|G| = \beta$ and $d(G) = \gamma < \beta$. Then G is β-resolvable.*

Proof: Let D be a dense subset of G such that $|D| = \gamma$. Consider

$$H = \langle D \rangle.$$

Then $|H| = \gamma$ and H is dense in G. This implies G is β-resolvable. Indeed, consider the left coset space G/H which has cardinality β and each coset $gH \in G/H$ is dense in G. $\square$

Consider a countable irresolvable topological space X. To the topological sum of uncountably many copies of X add a point z. The neighborhoods of z are all the summands with countably many exceptions. The resulting space Y is uncountable Lindelöf and irresolvable. However, for topological groups we have the following result.

Theorem 3.2. *Let G be a topological group, $|G| = \beta$, G is α-bounded and $\beta > \alpha$. Then G is resolvable.*

PROOF: We define a strictly increasing sequence $\{H_\nu : \nu \in \alpha^+\}$ of subgroups of G satisfying $|H_\nu| \leq \alpha$ for each $\nu < \alpha^+$.

If $\sigma < \alpha^+$ is a successor $\sigma = \rho + 1$, choose $K_\sigma \subseteq G \setminus H_\rho$, $|K_\sigma| = |\sigma|$, and let

$$H_\sigma = \langle K_\sigma \bigcup H_\rho \rangle.$$

If $\sigma < \alpha^+$ is a limit, choose $K_\sigma \subseteq G \setminus \bigcup_{\gamma \in \sigma} H_\gamma$, with $|K_\sigma| = |\sigma|$, and put

$$H_\sigma = \langle S_\sigma \cup \bigcup_{\gamma \in \sigma} H_\gamma \rangle.$$

Clearly $H_\gamma \subset H_\sigma$ for each $\gamma \in \sigma$. This completes our construction. Now we set $H = \bigcup_{\sigma \in \alpha^+} H_\sigma \subseteq G$, and $|H| > \alpha$. We shall prove that $H = \bigcup_{\sigma < \alpha^+} H_\sigma$ is resolvable. We define the following sets

$$L = \bigcup_{\nu \text{ limit}} (H_\nu \setminus \bigcup_{\gamma \in \nu} H_\gamma),$$

and

$$S = \bigcup_{\nu \text{ Successor}} (H_\nu \setminus \bigcup_{\gamma \in \nu} H_\gamma).$$

Obviously, $H = L \cup S$ and $L \cap S = \emptyset$. We claim that L and S are dense in H. First, H is not discrete because it is α-bounded and $|H| > \alpha$.

Suppose that S is not dense in H. Then there exists a non-empty open subset U of H with $U \subseteq L$. We have $U = qV$ for some $q \in H$, where V is a neighborhood of e in H, so $V = q^{-1}U$. There exists a subset $K \subseteq H$ with $|K| = \alpha < \beta$ such that

$$H = K \cdot V = K \cdot (q^{-1}U).$$

Obviously, $K \subseteq H_\gamma$ for some $\gamma \in \alpha^+$. We have $q^{-1} \in H_\lambda$ for some $\lambda \in \alpha^+$. Let $\rho = max\{\lambda, \gamma\}$, and choose $h \in H_{\rho+1} \setminus H_\rho$. There is $k \in K$ with $h = kq^{-1}u$, $u \in U$, and since k and q^{-1} are elements of $H_{\rho+1}$, we get

$$u = qk^{-1}h \in H_{\rho+1},$$

which contradicts the assumption $U \subseteq L$.

The same reasoning shows that L is dense in H. By Theorem 2.4 G is resolvable.

□

Corollary 3.3. *Every uncountable ω-bounded group G is resolvable.*

PROOF: Choose a subset $K \subseteq G$ of cardinality $\aleph_1$. Then define

$$H = \langle K \rangle.$$

The subgroup H has cardinality $\aleph_1$ and it is ω-bounded. Now apply Theorem 3.2 to obtain the resolvability of H and therefore of G. $\square$

Now we deduce some results about homogeneous spaces which contain convergent sequences. This will enable us to establish some constraints concerning irresolvable groups. A similar result has been noted independently by Comfort and García-Ferreira [CG] and by Comfort, Masaveau and Zhou [CMZ].

Lemma 3.4. *Let G be an infinite countable homogeneous space which contains a non-trivial convergent sequence. Then G is resolvable.*

PROOF: Let $\langle x_n \,|\, n \in \omega \rangle \subseteq G$ be a convergent sequence with the limit point x, and $G = \langle g_1, g_2, \dots \rangle$. For each $g_n \in G$, there exists a convergent sequence $A_{g_n} = \langle x_i^{g_n} \,|\, i \in \omega \rangle$, which has g_n as a limit point.

Let $\mathcal{B} = \langle B_i \,|\, i \in \omega \rangle$ be a family of infinite subsets of G such that each A_{g_n}, $n \in \omega$, occurs in this sequence infinitely many times.

Now construct finite subsets E_n, F_n of G by induction on n. Take $y_1, z_1 \in B_1$, $y_1 \neq z_1$ and put $E_1 = \{y_1\}$, $F_1 = \{z_1\}$. Suppose we have defined the sets

$$E_{n-1} = \{y_1, \dots, y_{n-1}\}, \qquad F_{n-1} = \{z_1, \dots, z_{n-1}\}.$$

Choose distinct elements y_n, z_n of the set $B_n \setminus (E_{n-1} \cup F_{n-1})$ and define

$$E_n = E_{n-1} \cup \{y_n\} \quad F_n = F_{n-1} \cup \{z_n\}.$$

This completes our construction. Now define

$$F = \bigcup_{n=1}^{\infty} F_n \qquad E = G \setminus F.$$

Then $G = E \cup F$ and we claim that both E and F are dense in G. Indeed, let U be an open set in G and $g_n \in U$. Almost all the points of A_{g_n} lie in U, so by the construction $U \cap E \neq \emptyset \neq U \cap F$. Thus G is resolvable. $\square$

Theorem 3.5. *Let G be an homogeneous space, which contains a convergent sequence. Then G is resolvable.*

PROOF: Let $X = \langle x_n : n \in \omega \rangle$ be a convergent sequence with limit point x, and $S = X \cup \{x\}$. We construct inductively an infinite sequence $\langle S_n : n \in \omega \rangle$ of subspaces of S. Let $S_1 = S$ and for each pair of points x_i, x_j of S_{n-1} there exists a homeomorphism $h_{i,j} : G \to G$ such that $x_i = h_{i,j}(x_j)$. Let K_{n-1} be the set of all such homeomorphisms $h_{i,j}$, for all $i, j \in \omega$.

Case 1. The group G has cardinality ω. This is given by Lemma 3.4.

Case 2. The group G is uncountable.

Let

$$S_n = \bigcup_{h_{i,j} \in G_{n-1}} h_{i,j}(S_{n-1})$$

and

$$S_\omega = \bigcup_{n \in \omega} S_n.$$

Clearly S_ω is a countable homogeneous subspace which contains a convergent sequence. By Theorem 3.4 S_ω is resolvable. For every $g \in G$ there exists a homeomorphism $h_g : G \to G$ with $h_g(x) = g$. Thus $G = \bigcup_{g \in G} h_g(S_\omega)$ and hence G is resolvable by Theorem 2.5. □

We now turn to irresolvable and maximal groups.

Theorem 3.6. ($p = \mathfrak{c}$) *Let G be a countable irresolvable topological group. Then $w(G) = \mathfrak{c}$.*

PROOF: Assume that $w(G) < \mathfrak{c}$. Consider a base $\mathcal{F}(e)$ of the filter of neighborhoods of the identity element with $|\mathcal{F}(e)| < \mathfrak{c}$. The family $\mathcal{F}(e)$ has the finite intersection property. By the principle $P = \mathfrak{c}$ we get a set $B \subseteq G$, $|B| = \aleph_0$, such that B converges to e. Therefore G is resolvable by Theorem 3.4, a contradiction. □

Corollary 3.7. ($p = \mathfrak{c}$) *If G is an irresolvable topological group, then $w(G) \geq \mathfrak{c}$.*

Corollary 3.8. ($p = \mathfrak{c}$) *Let G be a maximal topological group. Then $w(G) \geq \mathfrak{c}$.*

4. Questions

We end with some open problems.

1. Does there exist an irresolvable topological group which is not maximal?

A topological space is *extremally disconnected* if for every open set $U \subseteq X$, U^- is open.

Louveau [L] has shown that there exists under CH an extremally disconnected topological group. That group is resolvable. A natural question is therefore:

2. Does there exist an irresolvable topological group which is not extremally disconnected?

It was proved in [CGvM] that every pseudocompact topological group is resolvable. In [GG] spaces weakly-pseudocompact were defined as those spaces with the property of being G_δ-dense in some of their compactifications. It is then natural to ask:

3. Is every weakly-pseudocompact group resolvable?

The proof of Theorem 3.2 does not give β-resolvability of G.

4. Is every uncountable ω-bounded group β-resolvable for some infinite cardinal β? The referee has given an affirmative answer to this question for the case $\beta = \omega$.

5. Is it true that every α-bounded group of cardinality greater than α is resolvable? (See Theorem 3.3.)

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