On a condition weaker than insatiability condition

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Abstract. A condition weaker than the insatiability condition is given.

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An economy ε is defined by: m consumers indexed by $i = 1, 2, \dots, m$; n producers indexed by $j = 1, 2, \dots, n$; for each $i = 1, 2, \dots, m$ a consumption set $(X, \preceq_i)$, where X_i is a nonempty subset of $\boldsymbol{R}^\ell$ the production set for the producer j, and a priori vector $w \in \boldsymbol{R}^\ell$, called the total resources of ε. A state of economy ε is an $(m + n)$-tuple of $\boldsymbol{R}^\ell$, which can be represented by a point of $\boldsymbol{R}^{(m+n)\ell}$.

A state $(x, y) = ((x_i), (y_j))$ of ε is called attainable if $\sum_{i=1}^m x_i - \sum_{j=1}^m y_j = w$. The set of all attainable states of an economy ε will be denoted by A. An increasing function $u_i : X_i \to \boldsymbol{R}$ is called a utility function (i.e. $x_i, x_i' \in X_i$ with $x_i \preceq_i x_i' \Rightarrow u_i(x_i) \leq u_i(x_i')$).

In this note we consider the economy $\varepsilon = ((X_i, \preceq_{u_i}), (Y_j), w)$, where $X_i, \preceq_{u_i}, Y_j$ and w are defined as above, i.e. we are assuming that each preference preordering $\preceq_i$ can be represented by a utility function u_i. The utility function u_i is said to satisfy the insatiability condition if u_i has no greatest element with respect to $\preceq_{u_i}$. The greatest element of $\preceq_{u_i}$ is called a satiation consumption. Finally, a real valued function f defined on a convex set Y is said to be quasiconvex if for each real number t, the set $\{y \in Y : f(y) > t\}$ is either empty or convex.

Any other term or concept which is not defined here can be found in Debreu [1]. In [2] and [3] the author has proved the existence of Pareto optimum of an economy under the following condition (P) instead of insatiability condition:

(P) If $(x, y) = ((x_i), (y_j))$ and $(x', y') = ((x_i'), (y_j'))$ are two attainable states of an economy $\varepsilon = ((X_i, \preceq_{u_i}), (Y_i), w)$ such that $u_i(x_i) \geq u_i(x_i')$ for all i and $u_i(x_i) > u_i(x_i')$ for at least one i then there is an attainable state $(\overline{x}, \overline{y}) = ((\overline{x}_i), (\overline{y}_j))$ of ε such that $u_i(\overline{x}_i) > u_i(x_i')$ for each $i = 1, 2, \dots, m$.

The object of this note is to prove that under the usual conditions on the economy ε the condition (P) is weaker than the insatiability condition, i.e. the insatiability condition implies the condition (P). Thus the results proved in [2] is more general than the corresponding results of Debreu [1].

We first prove the following lemma.

Lemma 1. *If $(x,y)=((x_i),(y_j))$ and $(x',y')=((x'_i,y'_j))$ are two attainable states of an economy $\varepsilon=((X_i,\preceq_{u_i}),(Y_j),w)$, where each X_i is connected and no consumption is satiated and each u_i is continuous and if $u_i(x_i)\geq u_i(x'_i)$ for all i and $u_i(x_i)>u_i(x'_i)$ for at lest one i, then there is a state $(\overline{x},\overline{y})=((\overline{x}_i,\overline{y}_y))$ such that $u_i(\overline{x}_i)>u_i(x'_i)$ for each $i=1,2,\ldots,m$.*

PROOF: Let $J\subset\{1,2,\ldots,m\}$ such that $u_i(x_i)>u_i(x'_i)$ for all $i\in J$ and $K\subset\{1,2,\ldots,m\}$ such that $i\notin J$, i.e. $u_i(x)=u_i(x'_i)$ for all $i\in K$. Now we choose a number $\epsilon>0$ such that $\epsilon<\min\{u_i(x_i)-u_i(x'_i):i\in J\}$.

Since for each $i=1,2,\ldots,m$, X_i is connected and u_i is continuous and no consumption is satiated, it is possible to choose $\overline{x}=(\overline{x}_i)$ such that

$$u_i(\overline{x}_i)=\begin{cases} u_i(x'_i)+\frac{\epsilon}{s} & \text{if } i\in K;\\ u_i(x_i)+\frac{\epsilon}{r} & \text{if } i\in J, \text{ where } s \text{ and } r \text{ denote the cardinality}\\ & \text{of } K \text{ and } J \text{ respectively.}\end{cases}$$

Now it is clear that $u_i(\overline{x}_i)>u_i(x'_i)$ for each $i=1,2,\ldots,m$ and also for the sake of interest we note that

$$\begin{aligned}\sum_{i=1}^m \overline{u}_i(\overline{x}_i)&=\sum_{i\in K}u_i(\overline{x}_i)+\sum_{i\in J}u_i(\overline{x}_i)\\&=\sum_{i\in K}u_i(x'_i)+\epsilon+\sum_{i\in J}u_i(x_i)-\epsilon\\&=\sum_{i\in K}u_i(x_i)+\sum_{i\in J}u_i(x_i)=\sum_{i=1}^m u_i(x_i).\end{aligned}$$

□

Theorem 1. *Let $\varepsilon=((X_i,\preceq_{u_i}),(Y_j),w)$ be an economy such that*

(a) *for each $i=1,2,\ldots,m$*

 (i) *X_i is convex;*

 (ii) *u_i is continuous and quasiconcave;*

 (iii) *u_i is insatiable;*

(b) *$Y=\sum_{j=1}^n Y_j$ is convex.*

Then ε satisfies the condition (P).

PROOF: Let $(x,y)=((x_i),(y_j))$ and $(x',y')=((x'_i),(y'_j))$ be two attainable states of ε such that $u_i(x_i)\geq u_i(x'_i)$ for all i and $u_i(x_i)>u_i(x'_i)$ for at least one i. For each $i=1,2,\ldots,m$, let $O_i(x'_i)=\{\overline{x}_i\in X_i:u_i(\overline{x}_i)>u_i(x'_i)\}$. Then for each $i=1,2,\ldots,m$, $O_i(x'_i)$ is a nonempty open subset of X_i by virtue of the continuity of u_i and the Lemma.

In order to prove the theorem it will suffice to prove that $w \in \sum_{i=1}^m O_i(x_i') - Y$. We prove it by contradiction.

If possible, let $w \notin \sum_{i=1}^m O_i(x_i') - Y = Z$. Since by quasi concavity of u_i, $O_i(x_i')$ is convex and by (b) Y is convex, it follows that Z is convex. Hence by Minskowski's theorem (see Debreu [1, p. 25]) there is a hyperplane H through w bounding Z, i.e. there is $p \in \boldsymbol{R}^\ell$ such that $p \neq 0$ and $p \cdot a \geq p \cdot w$ for every $a \in Z$ where $\cdot$ is the inner product in $\boldsymbol{R}^\ell$. Now by the continuity of each u_i, it follows that $G = \sum_{i=1}^m C_i(x_i') - Y$ is contained in $C = \sum_{i=1}^m \overline{O_i(x_i')} - \sum_{j=1}^n Y_j$ where for each $i = 1, 2, \ldots, m$, $C_i(x_i') = \{\overline{x}_i \in X_i : u_i(\overline{x}_i) \geq u_i(x_i')\}$. Hence it follows that $\sum_{i=1}^m C_1(x_i') - Y$ is contained in C and hence in the closed half space above the hyperplane H. Now since $w = x' - y' \in G$, it minimizes $p \cdot a$ on G. Hence x_i' minimizes $p \cdot a$ on $C_1(x_i')$ for each i and $-y_j'$ minimizes $p \cdot a$ on $-Y_j$ (see e.g. Section 3.4 in [1, p. 45]). Hence by the result stated in [1, p. 93], $((x_i'), (y_j'))$ is an equilibrium with respect to the price p and by (6.3) in Debreu [1, p. 94], $((x_i'), (y_j'))$ is a Pareto optimum which is impossible. Hence $w \in Z$. □

References

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