Preservation and reflection of properties acc and hacc

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Abstract. The aim of the paper is to study the preservation and the reflection of acc and hacc spaces under various kinds of mappings. In particular, we show that acc and hacc are not preserved by perfect mappings and that acc is not reflected by closed (nor perfect) mappings while hacc is reflected by perfect mappings.

Keywords: acc space, hacc space, countably compact space, ordinal space, cofinality

Classification: 54D20, 54D30

Introduction and preliminary

Recently Matveev in ([5], [6]) introduced a new property acc (absolutely countably compact) which is stronger than countably compact and the property hacc (hereditarily absolutely countably compact); Matveev proved that the continuous image of an acc space need not to be acc and that the continuous pseudoopen image of an acc space is acc. Note in the Matveev's terminology a mapping $f : X \to Z$ is *pseudoopen* provided $\mathrm{Int}\,(f(O)) \neq \varnothing$ for every nonempty open set $O \subset X$; this property is stronger than the classical definition of "pseudoopen mapping" as given in [1], and [2]. In this paper we refer to Matveev's property as *strongly pseudoopen*.

We want to give answers concerning the preservation of acc and hacc spaces under various kinds of mappings; further we will show that the properties acc and hacc are not reflected by perfect mappings. Our counterexamples use product of ordinal spaces; the key of this problem is to know when a product of two ordinal spaces is acc or hacc. In the first part of this paper we give a solution to this problem by a characterization.

We will use the following definitions: X means a topological space; $\overline{A}$, $\mathrm{Int}\,(A)$ the closure and the interior of A respectively, where A is a subset of X; if $\mathrm{Int}\,(\overline{A}) = A$ ($\overline{\mathrm{Int}\,(A)} = A$) we say that A is a regular open (regular closed) subset of the space. A clopen subset of a topological space X is a set that is both open and closed subspace of X. $X_1 \oplus X_2$ denotes the discrete sum of disjoint spaces X_1 and X_2. Further a mapping $f : X \to Y$ is said to be closed (open) if for every closed (open) set $A \subset X$, the image $f(A)$ is closed (open); a mapping $p : X \to Y$ is

This research was supported by a grant from the C.N.R. (G.N.S.A.G.A.) and M.U.R.S.T. through "Fondi 40 %" Italy

called *perfect* if it is a continuous closed surjection and each fiber $p^{-1}(y)$, $(y \in Y)$ is compact. All mappings are assumed to be continuous.

Recall the following definitions:

Definition 1 [4]. *A space X is called countably compact provided every countable open cover has a finite subcover.*

Note that a characterization of countable compactness (see [4, 3.12.22 (d)]) states that a T_2 space is countably compact iff for every open cover $\mathcal{U}$ of X there exists a finite set $F \subset X$ such that $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \varnothing\} = X$.

Definition 2 [5]. *A space X is said to be absolutely countably compact* (*acc*) *provided for every open cover $\mathcal{U}$ of X and every dense $D \subset X$, there exists a finite set $F \subset D$ such that $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \varnothing\} = X$.*

Matveev ([5]) noted that:

$$\text{compact} \implies \text{acc} \overset{T_2}{\implies} \text{countably compact}$$

and he proved that every countably compact space X with countable tightness (see Definition 1.1) is acc. Further Vaughan ([10]) proved that every countably compact, orthocompact space (see Definition 1.2) is acc.

Matveev ([5]) proved that a regular closed subspace of an acc space need not be acc. Since every regular closed subspace of a topological space is a closed subspace, absolute countable compactness is a not hereditary with respect to closed subsets property. Then he introduced the following definition:

Definition 3 [5]. *A space X is hereditarily absolutely countably compact* (*hacc*) *if all closed subspaces of X are acc.*

1. Acc and hacc spaces as products of ordinal spaces

Definition 1.1 [4]. *If p is an element of a space X, the tightness of the point p in X is $t(p,X) = \min\{\kappa :$ for all $Y \subseteq X$ with $p \in \overline{Y}$, there is $A \subseteq Y$ with $|A| \leq \kappa$ and $p \in \overline{A}\}$; the tightness of the space X is $t(X) = \mathrm{Sup}\{t(p,X) : p \in X\}$.*

The following lemma is a basic result for the proof of the Theorem 1.2 that represent the key in the solution of the original problem.

Lemma 1.1. *Let α and β be ordinals. If $\omega \leq cf(\beta) < \alpha$, then $\alpha \times \beta$ is not acc.*

Proof: If $\omega = cf(\beta)$ we have that $\alpha \times \beta$ is not countably compact and then it is not acc. Suppose $\omega < cf(\beta)$ and consider the clopen subspace $(cf(\beta)+1) \times \beta \subset \alpha \times \beta$. Since all clopen subsets of an acc space are acc, we want to show that $(cf(\beta)+1) \times \beta$ is not acc. Put $cf(\beta) = \kappa$ and let $\{x_\delta : \delta < \kappa\}$ be a sequence such that $\{x_\delta : \delta < \kappa\}$ is cofinal in β and $\delta < \gamma \Leftrightarrow x_\delta < x_\gamma$. Consider the space $X = (\kappa+1) \times \beta$. Put $T = \{\kappa\} \times \beta$ and $D = X - T$. T is a closed subset of X and D is dense in X. For every $\delta < \kappa$ define $U_\delta = (\delta, \kappa] \times [0, x_\delta]$. Since $\{x_\delta : \delta < \kappa\}$

is cofinal in β, the family $\{U_\delta : \delta < \kappa\}$ is an open cover of T and then the family $\mathcal{U} = \{X - T\} \cup \{U_\delta : \delta < \kappa\}$ is an open cover of X. Suppose $F \subset D$ is finite. Denote $\xi = \max\{\eta \in (\kappa + 1) : (\eta, \gamma) \in F\}$ and consider the point $(\kappa, x_\xi) \in T$. For every $U \in \mathcal{U}$ such that $(\kappa, x_\xi) \in U$, we have that $U = U_\sigma$, where $\sigma \geq \xi$. Since every point $(\eta, \gamma) \in F$ is such that $\eta \leq \xi$, we have that $F \cap U_\sigma = \varnothing$, for every $\sigma \geq \xi$. So $(\kappa, x_\xi) \notin \mathrm{St}(F, \mathcal{U})$. By this, $\mathrm{St}(F, \mathcal{U}) \neq X$ for every finite $F \subset D$ which shows that X is not acc. □

Definition 1.2 [7]. *A space X is called orthocompact provided for every open cover $\mathcal{U}$ there exists an open refinement $\mathcal{V}$ such that for every $\mathcal{V}^| \subset \mathcal{V}$, we have $\bigcap\{V \in \mathcal{V}^| : x \in V\}$ is open for each $x \in X$.*

We prove a close analog of the following theorem of Brian Scott which will answer the question of when $\alpha \times \beta$ is acc or hacc.

Theorem 1.1 [7]. *Let α and β be ordinals. If $\alpha \leq \beta$, then the following are equivalent:*
1. *$\alpha \times \beta$ is orthocompact;*
2. *$\alpha \times \beta$ is normal;*
3. *one of the following holds:*
   (a) *$\alpha = \beta$ and $cf(\alpha) \leq \omega$;*
   (b) *$\alpha = \beta$ and $cf(\alpha) = \alpha$;*
   (c) *$cf(\alpha) \leq \omega$ and $\alpha \leq cf(\beta)$; and*
   (d) *$cf(\alpha) \leq \omega$ and $cf(\beta) \leq \omega$.*

The following is the main result of this section.

Theorem 1.2. *Let α and β be ordinals. If $\alpha \leq \beta$ and $\alpha \times \beta$ is countably compact, then the following are equivalent:*
1. *$\alpha \times \beta$ is acc;*
2. *$\alpha \times \beta$ is hacc;*
3. *$\alpha \times \beta$ is orthocompact;*
4. *$\alpha \times \beta$ is normal;*
5. *one of the following holds:*
   (a) *$\alpha = \beta$ and $cf(\alpha) < \omega$;*
   (b) *$\alpha = \beta$ and $cf(\alpha) = \alpha$;*
   (c) *$cf(\alpha) < \omega$ and $\alpha \leq cf(\beta)$; and*
   (d) *$cf(\alpha) < \omega$ and $cf(\beta) < \omega$.*

Proof: Clearly (5) in Theorem 1.2 implies (3) in Theorem 1.1; so for Theorem 1.2 we have (5) $\Rightarrow$ (4) $\Rightarrow$ (3). Since every countably compact, orthocompact space is acc ([10]) and orthocompactness and countable compactness are hereditary with respect to closed subsets, we have that (3) $\Rightarrow$ (2). As (2) $\Rightarrow$ (1), we have to show that (1) $\Rightarrow$ (5). The proof is by contradiction, and we note that by the hypothesis "$\alpha \times \beta$ is countably compact", we have $cf(\alpha) \neq \omega$ and $cf(\beta) \neq \omega$, however, α and β can be successor ordinals. Assume $\alpha = \beta$. If (a) and (b) do not hold, we have that $cf(\alpha) > \omega$ and $cf(\alpha) \neq \alpha$. Then $\omega < cf(\alpha) < \alpha$ and, by Lemma 1.1, $\alpha \times \alpha$

is not acc, a contradiction. Assume $\alpha \neq \beta$; therefore $\alpha < \beta$, by hypothesis. If (c) and (d) do not hold, we have that $(cf(\alpha) > \omega$ or $cf(\beta) < \alpha)$ and $(cf(\alpha) > \omega$ or $cf(\beta) > \omega)$ hold. This is equivalent to say that one of the following holds:

(i) $cf(\alpha) > \omega$ and $cf(\alpha) > \omega$, i.e. $cf(\alpha) > \omega$;
(ii) $cf(\alpha) > \omega$ and $cf(\beta) > \omega$;
(iii) $cf(\beta) < \alpha$ and $cf(\alpha) > \omega$;
(iv) $cf(\beta) < \alpha$ and $cf(\beta) > \omega$.

Since (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i), we have that $cf(\alpha) > \omega$ or $\omega < cf(\beta) < \alpha$ holds.

If $cf(\alpha) > \omega$, then $\omega < cf(\alpha) \leq \alpha < \beta$. So, by Lemma 1.1, $\beta \times \alpha$ is not acc. But $\beta \times \alpha$ is homeomorphic to $\alpha \times \beta$, a contradiction. If $\omega < cf(\beta) < \alpha$, by Lemma 1.1, $\alpha \times \beta$ is not acc, a contradiction. □

2. Preservation of property hacc under open and closed mappings

Recall the following proposition:

Proposition 2.1 [4]. *If $f : X \to Y$ is an open mapping, then for any subspace $L \subset Y$ the restriction $f_{|f^{-1}(L)} : f^{-1}(L) \to L$ is open.*

Proposition 2.2. *The hacc property is preserved by open mappings.*

Proof: Suppose $f : X \to Y$ is an open onto mapping and F a closed subset of Y. Since X is hacc and $f^{-1}(F)$ is closed in X, we have that $f^{-1}(F)$ is acc. Further the restriction $f_{|f^{-1}(F)}$ is open hence strongly pseudoopen; so the image $F = f^{-1}f(F)$ is acc. □

Proposition 2.3. *The hacc property is preserved by closed mappings.*

Proof: Suppose $f : X \to Y$ is a closed onto mapping, F is a closed subset of Y, $\mathcal{U}$ is an open cover of F and D a dense subset of F. Then $\mathcal{U}_0 = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of $f^{-1}(F)$. Since $\overline{f^{-1}(D)}$ is a closed subset of X and X is hacc, $\overline{f^{-1}(D)}$ is acc. $\mathcal{U}_0$ is an open cover of $\overline{f^{-1}(D)}$ because $\overline{f^{-1}(D)} \subset f^{-1}(F)$; as $f^{-1}(D)$ is a dense subset of $\overline{f^{-1}(D)}$, there exists A_0, a finite subset of $f^{-1}(D)$ such that $\mathrm{St}(A_0,\mathcal{U}_0) \supset \overline{f^{-1}(D)}$. Then $f(\mathrm{St}(A_0,\mathcal{U}_0)) \supset f(\overline{f^{-1}(D)})$. Since f is a closed surjection, F is a closed subset of Y and D is a dense subset of F, we have $f(\overline{f^{-1}(D)}) = F$ and hence that $\mathrm{St}(f(A_0),\mathcal{U}) = f(\mathrm{St}(A_0,\mathcal{U}_0)) \supset F$. This proves that Y is hacc. □

Corollary 2.1. *The hacc property is preserved by perfect mappings.*

3. Non-preservation and non-reflection of acc and hacc under various kinds of mappings

In this section we will give some examples to prove that the acc property is not preserved or reflected by perfect mappings; further we will show that the hacc property is not preserved by strongly pseudoopen mappings nor reflected by perfect mappings.

Recall the following proposition:

Proposition 3.1 [5]. *If $X = X_1 \cup \ldots \cup X_n$, X_i is acc for $1 \leq i \leq n$, and $X_i \subset \overline{\mathrm{Int}\,(X_i)}$ for $1 \leq i \leq n$ then X is acc.*

Now we show the following:

Example 3.1. *The acc property is not preserved by perfect mappings.*

Let us consider the space $H = H_1 \oplus H_2$, where $H_1 = \omega_2 \times (\omega_1 + 1)$ and $H_2 = \omega_2 \times (\omega_2 + 1)$. Now consider the factor mapping $h : H \to X$ which identifies the points of $\omega_2 \times \{\omega_1\} \subset H_1$ with the corresponding points of $\omega_2 \times \{\omega_2\} \subset H_2$. Now we show that the resulting factor space X is acc. Let $X_1 = h(H_1)$ and $X_2 = h(\omega_2 \times \omega_2)$ with $\omega_2 \times \omega_2 \subset H_2$. Note that the restriction of h to H_1 and of h to $\omega_2 \times \omega_2 \subset H_2$ are homeomorphisms, and that $X = X_1 \cup X_2$. Since, by Theorem 1.2, H_1 and $\omega_2 \times \omega_2$ are acc, X_1 and X_2 are acc. Let T be the image of the top-line of H_1 and H_2, i.e. $T = h(\omega_2 \times \{\omega_1\}) = h(\omega_2 \times \{\omega_2\})$; then $T \subset X_1$ and $T \cap X_2 = \varnothing$. $X_1 = h(H_1) \supset h(H_1) - T = X - (X_2 \cup T)$ that is open in X because $X_2 \cup T$ is closed in X (in fact $h^{-1}(X_2 \cup T)$ is closed in H). So $\mathrm{Int}\,(h(H_1)) \supset \mathrm{Int}\,(h(H_1) - T) = h(H_1) - T$; then $\overline{\mathrm{Int}\,(X_1)} = \overline{\mathrm{Int}\,(h(H_1))} \supset \overline{h(H_1) - T} = h(H_1) - \mathrm{Int}\,(T) = h(H_1) = X_1$. Since $h^{-1}(X_2)$ is open in H, X_2 is open in X and then $X_2 \subset \overline{\mathrm{Int}\,(X_2)}$. So, by Proposition 3.1 X is acc. In order to define a mapping with domain X, we identify $X_1 = \omega_2 \times (\omega_1 + 1) = H_1$ and $X_2 = h(\omega_2 \times \omega_2)$ with $\omega_2 \times \omega_2 \subset H_2$. Let $Z = \omega_2 \times (\omega_2 + 1)$. Then Z is not acc by Theorem 1.2. Define the mapping

$$\varphi : X \to Z$$

by $\varphi(\alpha, \beta) = (\alpha, \beta) \in (\omega_2 \times \omega_2)$, for all $(\alpha, \beta) \in X_2$ and $\varphi(\alpha, x) = (\alpha, \omega_2) \in (\omega_2 \times \{\omega_2\})$, for all $(\alpha, x) \in X_1$. Note that the restriction $\varphi|X_2$ is an homeomorphism (thus a continuous mapping) onto $\omega_2 \times \omega_2$, and the restriction $\varphi|X_1$ is the projection (thus a continuous mapping) of X_1 onto $\omega_2 \times \{\omega_1\}$ identified with $\omega_2 \times \{\omega_2\}$. Since $X = X_1 \cup X_2$, it is easy to check that φ is continuous, and since the projection $X \times Y \to X$ where Y is compact, is a closed mapping, it follows that φ is a closed mapping. Further each fiber $\varphi^{-1}(y)$, $(y \in \omega_2 \times (\omega_2 + 1))$ is compact and, by Theorem 1.2, Z is a non acc space; then φ is a perfect mapping from an acc space onto a non acc space and the proof is complete. □

Example 3.2. *The acc property is not reflected by perfect mappings.*

Consider the factor mapping $h : H \to X$ from Example 3.1; h is a continuous surjection such that each fiber $h^{-1}(x)$, $(x \in X)$ is compact (in fact, $|h^{-1}(x)| \leq 2$, for each $x \in X$). Further, as H_1 is homeomorphic to $h(H_1)$, H_2 is homeomorphic to $X_2 \cup T$ and $h(H_1)$ and $X_2 \cup T$ are closed subsets of X, every closed subset F of H is closed in X; then h is a closed mapping. So h is a perfect mapping onto an acc space such that $h^{-1}(X) = H$ is a non acc space (in fact, H_2 is a non acc, clopen subset of H). □

Example 3.3. *The hacc property is not preserved by strongly pseudoopen mapping.*

Consider the factor mapping $h : H \to X$ from Example 3.1. As the restriction of h to H_2 is an homeomorphism, $h(H_2)$ is a closed, non acc subspace of X; then X is a non hacc space. Let φ be the restriction of h to $[\omega_2 \times (\omega_1 + 1)] \oplus [\omega_2 \times \omega_2] \subset H$. By Theorem 1.2, $\omega_2 \times (\omega_1 + 1)$ and $\omega_2 \times \omega_2$ are hacc spaces, then $[\omega_2 \times (\omega_1 + 1)] \oplus [\omega_2 \times \omega_2]$ is hacc. Since h is a strongly pseudoopen mapping and the restriction of any strongly pseudoopen mapping to an open subset is strongly pseudoopen, we have that φ is a strongly pseudoopen (non open) mapping from an hacc space onto a non hacc space. □

Example 3.4. *The hacc property is not reflected by perfect mappings.*

Consider the factor space X from Example 3.1 and define the mapping

$$\varphi : X \to \omega_2 \times (\omega_1 + 1)$$

by $\varphi((\alpha, \beta)) = (\alpha, \beta) \in \omega_2 \times \omega_1$, for all $(\alpha, \beta) \in X_1 - T$ and $\varphi((\alpha, x)) = (\alpha, \omega_1) \in \omega_2 \times \{\omega_1\}$, for all $(\alpha, x) \in X_2 \cup T$. φ is a continuous mapping onto the hacc space $\omega_2 \times (\omega_1 + 1)$, by Theorem 1.2. Further each fiber $\varphi^{-1}(y)$, $(y \in \omega_2 \times (\omega_1 + 1))$ is compact because if $y \in \omega_2 \times \omega_1$, $|\varphi^{-1}(y)| = 1$ while if $y = (\alpha, \omega_1) \in \omega_2 \times \{\omega_1\}$, $\varphi^{-1}(y) = h(\{(\alpha, \omega_1)\} \cup \{\alpha\} \times (\omega_2 + 1))$, where $\{(\alpha, \omega_1)\} \subset H_1$ and $\{\alpha\} \times (\omega_2 + 1) \subset H_2$, i.e. $\varphi^{-1}(y)$ is a continuous image of a compact space, then it is a compact space. Further the mapping φ is closed (the proof is similar to the proof of Example 3.1). So φ is a perfect mapping onto an hacc space such that $\varphi^{-1}(\omega_2 \times (\omega_1 + 1)) = X$ is a non hacc space. □

4. Open questions

It is now natural to pose a question: is hacc preserved by strongly pseudoopen mappings with compact fibers?

Another natural question is the following: are acc and hacc reflected by perfect mappings (a) with first-countable fibers, (b) with sequential fibers, (c) with countably tight fibers.

Conclude the paper with the following diagram summing up all information we give concerning preservation and reflection of acc and hacc under various kinds of mappings.

Table

PRESERVATION AND REFLECTION OF ACC AND HACC

TYPES OF PROPERTIES	TYPES OF MAPPINGS			
	open	closed	perfect	strongly pseudoopen
preservation of acc	+	−	−	+*
preservation of hacc	+	+	+	−
reflection of acc	?	−	−	?
reflection of hacc	?	−	−	?

(*) See ([5]).

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(*Received* February 21, 1995, *revised* September 11, 1995)