On nodal radial solutions of an elliptic problem involving critical Sobolev exponent

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Abstract. In this paper we construct radial solutions of equation (1) (and (13)) having prescribed number of nodes.

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1. Introduction

In this note we study the existence of nodal radial solutions of the problem

$$-\triangle u = K(|x|)\,|u|^{2^*-2}\,u \quad \text{in} \quad \mathbb{R}_n, \tag{1}$$

where $2^* = \frac{2n}{n-2}$, $n \geq 3$. It is known that under suitable assumptions on K, problem (1) has infinitely many radial solutions (see for example [4] and references given there). Recently, Bartsch and Willem [3] showed that the problem

$$-\triangle u + b(|x|)u = f(|x|\,,\,u) \quad \text{in} \quad \mathbb{R}_n, \tag{2}$$

with f having a subcritical growth, admits for each integer $k \geq 0$, a pair of radial solutions u_k^+ and u_k^- with $u_k^-(0) < 0 < u_k^+(0)$. This result gives the existence of infinitely many nodal and radial solutions. We show that this result continues to hold for problem (1). The paper is organized as follows. In Section 2 we consider a constrained minimization and min-max procedure of the mountain-pass type for potential operator equations in a reflexive Banach space. We show that both minimization levels are the same. This information will be used in Section 3 to construct a solution of (1) by solving the Dirichlet problem in a ball and in an annulus. We then represent $\mathbb{R}_n$ as a union of a ball and a finite number of annuli, the last one of which is the exterior of a ball. We associate with this representation of $\mathbb{R}_n$ a sum of variational problems, which give a rise to the existence of nodal solutions.

It is worth noting that the underlying Sobolev space for problem (1) is $D^{1,2}(\mathbb{R}_n)$ obtained as a completion of $C_0^\infty(\mathbb{R}_n)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}_n} |Du(x)|^2\,dx.$$

By the Sobolev embedding theorem, we have the continuous embeddings $H^1(\mathbb{R}_n) \subset D^{1,2}(\mathbb{R}_n) \subset L^{2^*}(\mathbb{R}_n)$. This slightly contrasts, with a situation in paper [3]. It is assumed there that b is continuous and bounded from below by a positive constant. Consequently, the underlying Sobolev space for problem (2) is $E =$ the completion of $C_0^\infty(\mathbb{R}_n)$ with respect to the norm

$$\|u\|_E^2 = \int_{\mathbb{R}_n} (|Du|^2 + b(|x|)u^2)\, dx$$

and we have the continuous embeddings $E \subset H^1(\mathbb{R}_n) \subset L^{2^*}(\mathbb{R}_n)$.

2. Some results on potential operator equations in reflexive Banach spaces

We commence with some basic definitions and notations. Let X be a reflexive Banach space. The dual space X is denoted by X^*. We denote the duality pairing between X and X^* by $\langle \cdot, \cdot \rangle$. The weak convergence in X and X^* is denoted by "$\rightharpoonup$" and the strong convergence by"$\to$". For functionals $a : X \to \mathbb{R}$ and $b\colon X \to \mathbb{R}$ we use letters A and B to denote their Fréchet or Gâteaux derivatives.

The following terminology is standard and can be found in the monograph by Vainberg [10].

A mapping $A : X \to X^*$ is said to be a potential operator with a potential $a : X \to \mathbb{R}$, if a is Gâteaux differentiable and

$$\lim_{t\to 0} t^{-1}(a(u + tv) - a(u)) = \langle A(u), v\rangle$$

for all u, v in X. For a potential a we always assume that $a(0) = 0$.

A mapping $A : X \to X^*$ is called hemicontinuous if it is continuous on line segments in X and with X^* equipped with the weak topology.

A potential a of a hemicontinuous potential operator A can be represented in the form

$$a(u) = \int_0^1 \langle A(tu), u\rangle\, dt$$

for all $u \in X$.

We say that mapping $A\colon X \to X^*$ is homogeneous of degree $\beta > 0$, if for every $u \in X$ and $t > 0$

$$A(tu) = t^\beta A(u).$$

Consequently, for the hemicontinuous and homogeneous potential operator A of degree β with the potential a, we have

$$a(u) = \frac{1}{\beta + 1}\langle A(u), u\rangle.$$

A mapping $A:\ X \to X^*$ is strongly monotone if there exists a continuous function $\mathcal{H}:\ [0,\infty) \to [0,\infty)$ which is positive on $(0,\infty)$ and $\lim_{t\to\infty} \mathcal{H}(t) = \infty$ and such that

$$\langle A(u) - A(v),\, u - v\rangle \geq \mathcal{H}(\|u-v\|)\,\|u-v\|\,.$$

A mapping $A:\ X \to X^*$ is said to satisfy the condition S_1, if for every sequence $\{u_j\} \subset X$ with $u_j \rightharpoonup u$ and $A(u_j) \to v$ in X^* we have $u_j \to u$ in X. Evidently, every strongly monotone operator satisfies the condition S_1.

We consider the potential operator equation

$$Au = Bu \quad \text{in} \quad X, \tag{3}$$

where $A:\ X \to X^*$ and $B:\ X \to X^*$ are potential operators with potentials $a:\ X \to \mathbb{R}$ and $b:\ X \to \mathbb{R}$, respectively.

Throughout this section we make the following assumptions.

(A_1) The mapping $A:\ X \to X^*$ is a homogeneous continuous potential operator of degree $p-1$, $p>1$, with a potential a. Moreover, we assume that

$$k_1 \leq a(u) \leq k_2 \tag{4}$$

for all u with $\|u\| = 1$ and some positive constants k_1 and k_2.

Since the potential a is homogeneous of degree p, we see that (4) implies that

$$k_1\,\|u\|^p \leq a(u) \leq k_2\,\|u\|^p \tag{5}$$

for all $u \in X$. Also, the continuity of the potential operator A implies the Fréchet differentiability of its potential a.

(A_2) $b \in C^1(X,\mathbb{R})$ and we assume that there exists an $\alpha > p$ such that

$$\langle B(u),\, u\rangle \geq \alpha b(u) > 0 \quad \text{on} \quad X - \{0\}.$$

(A_3) A potential $\tilde{b}:\ X \to \mathbb{R}$ defined by

$$\tilde{b}(u) = \langle B(u),\, u\rangle - pb(u)$$

satisfies the following growth condition

$$\tilde{b}(u) \leq K(\|u\|^\alpha + \|u\|^r)$$

for all $u \in X$ and some constants $K > 0$ and $r > \alpha$.

We list the immediate consequences of the hypotheses (A_2) and (A_3).

We put for a fixed $u \in X - \{0\}$

$$h(t) = t^{-\alpha} b(tu) \quad \text{for} \quad t > 0.$$

Since

$$h'(t) = \frac{\langle B(tu),\, tu\rangle - \alpha b(tu)}{t^{\alpha+1}} \geq 0$$

for $t > 0$, we have

(6) $$b(tu) \leq t^{\alpha} b(u) \quad \text{for} \quad 0 < t \leq 1,$$

and

(7) $$b(tu) \geq b(u) t^{\alpha} \quad \text{for} \quad t > 1.$$

In particular, $b(0) = 0$ and $B(0) = 0$.

If $u \neq 0$, then $t^{-p} b(tu)$ is strictly increasing for $t > 0$. Indeed, we have

$$\frac{d}{dt}(t^{-p} b(tu)) = \frac{\langle B(tu),\, tu\rangle - pb(tu)}{t^{p+1}} > 0.$$

It follows from (A_2) and (A_3) that

(8) $$b(u) \leq \frac{K}{\alpha - p}\left(\|u\|^{\alpha} + \|u\|^{r}\right)$$

and

(9) $$\langle B(u),\, u\rangle \leq \frac{\alpha K}{\alpha - p}(\|u\|^{\alpha} + \|u\|^{r})$$

for all $u \in X - \{0\}$.

We now define a variational functional $F :\, X \to \mathbb{R}$ by

$$F(u) = a(u) - b(u).$$

Any critical point of F is a solution of (3). Our main existence result is based on the Ambrosetti-Rabinowitz mountain pass theorem [2]. Towards this end, we put

(10) $$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} F(g(t)),$$

where $\Gamma = \{g \in C([0,\, 1],\, X)\,;\, g(0) = 0 \ \text{ and } \ F(g(1)) < 0\}$.

In Theorem 1 below we shall show that under additional assumptions on B, c is a critical value of F.

Theorem 1. *Suppose that A is strongly monotone potential operator and that B is strongly continuous, that is, $u_m \rightharpoonup u$ in X implies $B(u_m) \to B(u)$ in X^*. Then the equation* (3) *has a nontrivial solution.*

PROOF: It is easy to check the assumptions of the mountain pass theorem from [2]. It follows from (5) and (8) that

$$F(u) \geq k_1 \|u\|^p - \frac{K}{\alpha - p} \left(\|u\|^\alpha + \|u\|^r\right).$$

Since $p < \alpha < r$, there exist $\delta > 0$ and $\rho > 0$ such that $F(u) \geq \rho$ for $\|u\| = \delta$ and also $F(u) > 0$ on $0 < \|u\| \leq \delta$. Let $\|\overline{u}\| > \delta$, by (5) and (7) we have for $t \geq 1$ that

$$F(t\overline{u}) \leq k_2 t^p \|\overline{u}\|^p - t^\alpha b(\overline{u}) < 0$$

for $t > 1$ sufficiently large. We now show that F satisfies the Palais-Smale condition, that is, if $\{u_m\} \subset X$ is such that $\{F(u_m)\}$ is bounded and $F'(u_m) \to 0$ in X^*, then $\{u_m\}$ possesses a convergent subsequence. It is easy to check that $\{u_m\}$ must be bounded and we may assume that $u_m \rightharpoonup u$. Since $F'(u_m) = A(u_m) - B(u_m) \to 0$ and $B(u_m) \to B(u)$ in X^*, we see that $A(u_m)$ is convergent in X^*. According to condition S_1, $u_m \to u$ in X. Applying the mountain pass theorem we see that c is a critical value of F. Since $c \geq \rho$, a critical point corresponding to c must be non-zero. □

Any non-zero critical point of F belongs to Nehari's manifold (the set of artificial constraints) which is defined as follows

$$\mathcal{N} = \{u \in X - \{0\}\,;\, \langle A(u), u\rangle = \langle B(u), u\rangle\}.$$

It follows from (5) and (9) that

$$pk_1 \|u\|^p \leq \frac{\alpha K}{\alpha - p} \left(\|u\|^\alpha + \|u\|^r\right)$$

for all $u \in \mathcal{N}$. Since $p < \alpha < r$, we see that there exists a constant $d > 0$ such that

$$\mathcal{N} \subset X - B(0, d), \tag{11}$$

where $B(0, d)$ is a ball of radius d centered at 0. In general, the set $\mathcal{N}$ is unbounded. If $u \in \mathcal{N}$, then

$$pF(u) = pa(u) - pb(u) = \langle B(u), u\rangle - pb(u) = \tilde{b}(u).$$

On the other hand we have by (A_2)

$$\begin{aligned}\tilde{b}(u) &= \langle B(u), u\rangle - pb(u) \geq (1 - \frac{p}{\alpha})\langle B(u), u\rangle \\ &= (1 - \frac{p}{\alpha})\langle A(u), u\rangle \geq k_1(1 - \frac{p}{\alpha}) \|u\|^p.\end{aligned}$$

This combined with (11) implies that

$$M = \inf\{F(u)\,;\, u \in \mathcal{N}\} > 0.$$

Since F is only C^1, $\mathcal{N}$ is not a differentiable manifold and we cannot apply a standard Lagrange multiplier method as in [9]. However, we will be able to show that $M = c$ and every $u_0 \in \mathcal{N}$ such that $F(u_0) = M$ is a critical point of F.

Lemma 1. *Suppose that $t^{-p}\langle B(tu),\, tu\rangle$ is strictly increasing function on $(0,\, \infty)$ for each $u \neq 0$. Then, there exists a unique continuous function $t :\, X - \{0\} \to (0,\, \infty)$ such that:*

if $u \in X - \{0\}$ and $s > 0$, then $su \in \mathcal{N}$ if and only if $s = t(u)$.

In particular, $t(u) = 1$ if and only if $u \in \mathcal{N}$.

Proof: Let $u \in X - \{0\}$ and we put for $t > 0$

$$\psi(t) = \langle A(u),\, u\rangle - t^{-p}\langle B(tu),\, tu\rangle.$$

The function ψ is continuous on $(0,\, \infty)$. By (9) $\lim_{t\to 0} \psi(t) = \langle A(u),\, u\rangle$ and by virtue of (A_2) and (7) $\lim_{t\to\infty} \psi(t) = -\infty$. Since $t^{-p}\langle B(tu),\, tu\rangle$ is strictly increasing on $(0,\, \infty)$ there exists a unique $t(u) > 0$ such that $t(u) \cdot u \in \mathcal{N}$. To show the continuity of $t(u)$, let $u_m \to u$ in $X - \{0\}$. By virtue of (A_2) and (7) we have for $t_m = t_m(u_m) \geq 1$

$$\begin{aligned} k_2 t_m^p \|u_m\|^p &\geq \langle A(t_m u_m),\, t_m u_m\rangle = \langle B(t_m u_m),\, t_m u_m\rangle \\ &\geq \alpha b(t_m u_m) \geq \alpha t_m^\alpha b(u_m). \end{aligned}$$

Since $u \neq 0$, we may assume that $b(u_m) \geq \delta > 0$ for some $\delta > 0$ and all m. This, combined with the previous estimate, implies that $\{t_m(u_m)\}$ is a bounded sequence. If a subsequence of $\{t_m\}$ converges to t_0, then, since $t_m \cdot u_m \in \mathcal{N}$ for each m, we must have that $t_0 \cdot u \in \mathcal{N}$ and hence $t_0 = t(u)$. This obviously means that $t(u_m) \to t(u)$. □

Lemma 2. *If $t^{-p}\langle B(tu),\, tu\rangle$ is strictly increasing function on $(0,\, \infty)$, then*

$$M = c = \inf_{u\in X-\{0\}} \max_{t\geq 0} F(t\cdot u).$$

Proof: Let $D = \inf_{u\in X-\{0\}} \max_{t\geq 0} F(tu)$. It is obvious that

$$M \leq F(t(u)u) = \max_{t\geq 0} F(t\cdot u)$$

and this implies that $M \leq D$. If $u \in \mathcal{N}$, then $F(u) = \max_{t\geq 0} F(tu)$. Hence,

$$M = \inf_{u\in\mathcal{N}} \max_{t\geq 0} F(t\cdot u) \geq \inf_{u\in X-\{0\}} \max_{t\geq 0} F(tu) = D$$

and we have $M = D$. Since $F(t \cdot u) < 0$ for t large, we must have that $c \leq D$. Finally, we observe that each $g \in \Gamma$ must intersect $\mathcal{N}$, so $c \geq M$. □

We now show that if $u_0 \in \mathcal{N}$ and $F(u_0) = M$, then u_0 is a critical point of F. A similar result for equation (2) has been established in [3] by means of the quantitative deformation lemma. Here, we give an elementary proof of this result without the use of a deformation lemma and in which we use an idea from the paper [1]. □

Proposition 1. *Suppose that $t^{-p}\langle B(tu), u\rangle$, $u \neq 0$, is strictly increasing function on $(0, \infty)$. Then for every $u_0 \in \mathcal{N}$ such that $F'(u_0) \neq 0$ we have*

$$F(u_0) > M = \inf\{F(u)\,;\, u \in \mathcal{N}\}.$$

In particular, if $M = F(u_0)$, $u_0 \in \mathcal{N}$, then $F'(u_0) = 0$.

Proof: Let $u_0 \in \mathcal{N}$ be such that $F'(u_0) \neq 0$. We choose $h_0 \in X$ such that $\langle F'(u_0), h_0\rangle = 1$ and we put $\sigma_t(\alpha) = \alpha u_0 - t h_0$. Then

$$\lim_{\substack{t\to 0\\ \alpha\to 1}} \frac{d}{dt} F(\sigma_t(\alpha)) = \langle F'(u_0), h_0\rangle = -1.$$

We can find $\varepsilon > 0$ and $\delta > 0$ such that for all $\alpha \in [1-\varepsilon, 1+\varepsilon]$ and $t \in [0, \delta]$ we have

$$F(\sigma_t(\alpha)) < F(\sigma_0(\alpha)) = F(\alpha u_0). \tag{12}$$

We put

$$H_t(\alpha) = pa(\sigma_t(\alpha)) - \langle B(\sigma_t(\alpha)), \sigma_t(\alpha)\rangle.$$

Since $u_0 \in \mathcal{N}$, we have

$$H_0(1) = pa(u_0) - \langle B(u_0), u_0\rangle = 0.$$

On the other hand, using the fact that $t^{-p}\langle B(tu_0), tu_0\rangle$ is an increasing function on $(0, \infty)$ and taking $\varepsilon > 0$ and $\delta > 0$ smaller if necessary, we may assume that for $0 < t \leq \delta$

$$H_t(1-\varepsilon) < 0 \quad\text{and}\quad H_t(1+\varepsilon) > 0.$$

Hence, for each $t \in (0, \delta]$ there exists $\alpha_t \in [1-\epsilon, 1+\epsilon]$ such that $H_t(\alpha_t) = 0$. This means that $\sigma_t(\alpha_t) \in \mathcal{N}$. Consequently, it follows from (12) that

$$\begin{aligned} M &= \inf\{F(u)\,;\, u \in \mathcal{N}\} \leq F(\sigma_t(\alpha_t)) < F(\alpha_t u_0) \\ &\leq \sup_{t\geq 0} F(tu_0) = F(u_0), \end{aligned}$$

and this completes the proof. □

3. Radial solutions

We now shift our attention to problem (1). We write it in a more general form

$$-\Delta u = K(|x|)f(u) \quad \text{in} \quad \mathbb{R}_n, \tag{13}$$

where K and f are continuous functions on $[0, \infty)$ and $\mathbb{R}$, respectively, and satisfying the following conditions

(a) $f(u)u \geq \mu \int_0^u f(s)\,ds$, $|f(u)| \leq a_1 |u|^{2^*-1}$ for some constants $2 < \mu < 2^*$ and $a_1 > 0$ and all $u \in \mathbb{R}$,

(b) $K(0) = 0$, $0 = K(\infty) = \lim_{|x|\to\infty} K(|x|)$ and $K(|x|) > 0$ on $\mathbb{R}_n - \{0\}$,

(c) $\frac{f(u)}{|u|}$ is a strictly increasing function of $u \in \mathbb{R} - \{0\}$.

Following Bartsch-Willem [3] we put for $0 \leq \rho < \sigma \leq \infty$

$$\Omega(\rho, \sigma) = \operatorname{Int}\{x \in \mathbb{R}_n\,;\ \rho \leq |x| < \sigma\}.$$

Thus $\Omega(0, \sigma)$ is a ball, $\Omega(0, \infty) = \mathbb{R}_n$, $\Omega(\rho, \sigma)$ is an annulus if $0 < \rho < \sigma < \infty$ and $\Omega(\rho, \infty)$ is the exterior of a ball.

We now consider the Dirichlet problem

$$\begin{cases} -\Delta u = K(|x|)f(u) & \text{in } Q, \\ u(x) = 0 & \text{on } \partial Q, \end{cases} \tag{14}$$

where Q is either $\Omega(0, \sigma)$, $0 < \sigma < \infty$, or $\Omega(\rho, \sigma)$, $0 < \rho < \sigma < \infty$ or $\Omega(\rho, \infty)$, $0 < \rho < \infty$. If $\sigma = \infty$, that is, either the case of $\mathbb{R}_n$ or the exterior of a ball, we denote by $D_r^{1,2}(\Omega(\rho, \infty))$ the subspace of all radially symmetric functions in $D^{1,2}(\Omega(\rho, \infty))$, where the definition of the space $D^{1,2}(\mathbb{R}_n)$ has been given in Section 1 and the space $D^{1,2}(\Omega(\rho, \infty))$, $0 < \rho < \infty$, is defined in the same manner. Finally, by $W_{0,r}^{1,2}(\Omega(\rho, \sigma))$, $0 \leq \rho < \sigma < \infty$, we denote the set of all radially symmetric functions in $W_0^{1,2}(\Omega(\rho, \sigma))$. Obviously, the spaces $D_r^{1,2}(\Omega(\rho, \infty))$ and $W_{0,r}^{1,2}(\Omega(\rho, \sigma))$ are equipped with norms from the spaces $D^{1,2}(\Omega(\rho, \infty))$ and $W_0^{1,2}(\Omega(\rho, \sigma))$, respectively. For the ease of notation we set

$$E(Q) = \begin{cases} W_{0,r}^{1,2}(Q), & \text{if } Q = \Omega(\rho, \sigma),\ 0 \leq \rho < \sigma < \infty, \\ D_r^{1,2}(Q), & \text{if } Q = \Omega(\rho, \infty),\ 0 < \rho < \infty. \end{cases}$$

To solve the Dirichlet problem (14) we consider a variational functional $\mathcal{F} : E(Q) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \frac{1}{2}\int_Q |Du|^2\,dx - \int_Q K(|x|)F(u)\,dx,$$

where $F(u) = \int_0^u f(s)\,ds$. Any critical point of $\mathcal{F}$ is a solution to problem (14). As in Section 1 we define a potential $b:\ E(Q) \to \mathbb{R}$, by

$$b(u) = \int_Q K(|x|)F(u)\,dx.$$

Its potential operator is given by

$$\langle B(u),\, v\rangle = \int_Q K(|x|)f(u)v\,dx$$

for all $u, v \in E(Q)$. We show that B is strongly continuous from $E(Q)$ into its dual space $E(Q)^*$, that is, $u_m \rightharpoonup u$ in $E(Q)$ implies $B(u_m) \to B(u)$ in $E(Q)^*$. Since $W^{1,2}_{0,r}(\Omega(\rho,\,\sigma))$, $0 < \rho < \sigma < \infty$ is compactly embedded into $L^{2^*}(\Omega(\rho,\,\sigma))$ (see for example [4, Proposition 8]) it suffices to consider cases: (i) $Q = \Omega(0,\,\sigma)$, $0 < \sigma < \infty$, and (ii) $Q = \Omega(\rho,\,\infty)$, $0 < \rho < \infty$. Extending every function in $E(Q)$ by 0 outside Q we can consider $E(Q)$ as a subspace of $D^{1,2}_r(\mathbb{R}_n)$. According to Lemma 1 in [8] every function u in $E(Q)$ is almost everywhere equal to a continuous function U for $x \neq 0$ such that

$$|U(x)| \le (\omega_n(n-2))^{-\frac{1}{2}}\,|x|^{\frac{2-n}{2}}\,\|Du\|_2\,, \tag{15}$$

where ω_n is a volume of a unit ball in $\mathbb{R}_n$ (see also [6]). Let $u_m \rightharpoonup u$ in $W^{1,2}_{0,r}(\Omega(0,\,\sigma))$, $0 < \sigma < \infty$. By the Sobolev embedding theorem we may assume that $u_m \longrightarrow u$ a.e. on $\Omega(\rho,\,\sigma)$. Let $v \in W^{1,2}_{0,r}(\Omega(0,\,\sigma))$ with $\|v\| \le 1$, then for $0 < \delta < \sigma$, we write

$$\langle B(u_m),\, v\rangle - \langle B(u),\, v\rangle = \int_{\Omega(0,\delta)} K(|x|)(f(u_m) - f(u))v\,dx \\ + \int_{\Omega(\delta,\sigma)} K(|x|)(f(u_m) - f(u))\cdot v\,dx = J_1 + J_2.$$

By the Hölder and Sobolev inequalities we have

$$|J_1| \le a_1 S^{-1} \sup_{\Omega(0,\delta)} K(|x|) \left[\sup_{m\ge 1}\left(\int_{\Omega(0,\sigma)} |u_m|^{2^*}\,dx\right)^{\frac{2^*-1}{2^*}} + \right. \\ \left. + \left(\int_{\Omega(0,\sigma)} |u|^{2^*}\,dx\right)^{\frac{2^*-1}{2^*}}\right].$$

Since $K(0) = 0$, given $\varepsilon > 0$ we can find $\delta < \sigma$ such that $|J_1| < \varepsilon$ for all $m \ge 1$. Due to the uniform bound of u_m on $\Omega(\delta,\,\sigma)$ given by (15) and the Lebesgue dominated convergence theorem we see that $\lim_{m\to\infty} J_2 = 0$ uniformly in $\|v\| \le 1$. The proof of the case (ii) is similar and is omitted.

It is clear that b and B satisfy hypotheses (A_1), (A_2) and (A_3) of Section 1, $t^{-2}\langle B(tu), tu\rangle$, $u \ne 0$, is strictly increasing in $t > 0$ and consequently we can formulate the existence result for problem (14). However, to get a nodal solution of problem (13) we need to show that problem (14) has a positive and negative solution in $E(Q)$. This will be achieved by modifying the nonlinearity f.

Let

$$g(u) = \begin{cases} f(u) & \text{for } u \ge 0, \\ -f(-u) & \text{for } u < 0 \end{cases}$$

and $G(u) = \int_0^u g(s)\,ds$. Since g satisfies assumptions (a), (b) and (c) we can apply Theorem 1 and Lemma 2 to the functional $\mathcal{F}_+(u) = \frac{1}{2}\int_Q |Du|^2\,dx - \int_Q K(|x|)G(u)\,dx$. As a result we obtain the existence of a nontrivial solution u^+ of problem (14) given by

$$\mathcal{F}_+(u^+) = c_+ = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} \mathcal{F}_+(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E(Q));\ \gamma(0) = 0 \text{ and } \mathcal{F}_+(\gamma(1)) < 0\}$. Obviously, $u^+ \in \mathcal{N}_+ = \{u \in E(Q) - \{0\};\ \langle \mathcal{F}_+'(u), u\rangle = 0\}$ and by Lemma 2 we have

$$c_+ = M_+ = \inf_{u\in\mathcal{N}_+} \mathcal{F}_+(u).$$

Since g is odd, G must be even. Hence $\mathcal{F}_+(u^+) = \mathcal{F}_+(|u^+|)$ and we may assume that $u^+ \ge 0$ on Q. The maximum principle implies that $u^+ > 0$ on Q.

Similarly, we put

$$h(u) = \begin{cases} -f(-u) & \text{for } u \ge 0, \\ f(u) & \text{for } u < 0, \end{cases}$$

$H(u) = \int_0^u h(s)\,ds$, $\mathcal{F}_-(u) = \frac{1}{2}\int_Q |Du|^2\,dx - \int_Q K(|x|)H(u)\,dx$ and $\mathcal{N}_- = \{u \in E(Q) - \{0\};\ \langle \mathcal{F}_-'(u), u\rangle = 0\}$. If follows from Theorem 1 and Lemma 2 that problem (14) has a negative solution u^- satisfying

$$\mathcal{F}_-(u^-) = c_- = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} \mathcal{F}_-(\gamma(t))$$

and

$$\mathcal{F}_-(u^-) = M_- = \inf_{u\in\mathcal{N}_-} \mathcal{F}_-(u).$$

Summarizing, we can formulate the existence result:

Proposition 2. *The Dirichlet problem* (14) *admits a positive and negative solutions* u^+ *and* u^- *such that* $\mathcal{F}(u^+) = M_+$ *and* $\mathcal{F}(u^-) = M_-$.

Since $Q = \Omega(\rho, \sigma)$, we write, $M_\pm = M_\pm(\rho, \sigma)$, $\mathcal{N}_\pm = \mathcal{N}_\pm(\rho, \sigma)$ and in Proposition 3 below we examine the behaviour of $M_\pm$ as a function of ρ and σ. We write $M(\rho, \sigma)$ for both $M_-(\rho, \sigma)$ and $M_+(\rho, \sigma)$ and $\mathcal{N}$ for $\mathcal{N}_-(\rho, \sigma)$ and $\mathcal{N}_+(\rho, \sigma)$.

Proposition 3. $M(\rho, \sigma)$ *has the following properties:*

(i) *if* $0 \le \rho \le \rho' < \sigma' \le \sigma \le \infty$, *then* $M(\rho, \sigma) \le M(\rho', \sigma')$

(ii) $\lim\limits_{\sigma-\rho\to 0} M(\rho, \sigma) = \infty$,

(iii) $\lim\limits_{\rho\to\infty} M(\rho, \infty) = \infty$,

(iv) $M(\rho, \sigma)$ *is lower semicontinuous.*

Proof: (i) follows from the fact that $\mathcal{N}(\rho', \sigma') \subset \mathcal{N}(\rho, \sigma)$.

(ii) We need the following estimate: let $Q \subset \mathbb{R}_n$ be a bounded domain, then for every $u \in W_0^{1,2}(Q)$ we have

$$\|u\|_q \le c\,|Q|^{\frac{1}{q}-\frac{1}{2^*}}\,\|Du\|_2 \tag{16}$$

for all $2 \le q \le 2^*$ and some positive constant c. If $q = 2^*$, then $c = S^{-1}$ (the best Sobolev constant) (see [5] p.45). We distinguish two cases:

$$(\alpha)\quad \sigma-\rho\to 0,\quad \rho\to\delta>0, \qquad (\beta)\quad \sigma-\rho\to 0,\quad \rho\to 0.$$

Let $u \in \mathcal{N} = \mathcal{N}(\rho, \sigma)$. In the first case it follows from (15) and (16) that

$$\begin{aligned}
\int_{\Omega(\rho,\sigma)} |Du|^2\,dx &= \int_{\Omega(\rho,\sigma)} K(|x|)F(u)\,dx\\
&\le C(n)\rho^{\frac{2-n}{2}\beta}\,\|Du\|_2^{\beta}\int_{\Omega(\rho,\sigma)} |u|^{2^*-\beta}\,dx\\
&\le C_1(n)\rho^{\frac{2-n}{2}\beta}\,(\sigma^n-\rho^n)^{\frac{1}{2^*-\beta}-\frac{1}{2^*}}\,\|Du\|_2^{2^*},
\end{aligned}$$

where $2 < \beta < 2^*$ and $C(n)$, $C_1(n)$ are positive constants independent of u, ρ and σ. Consequently, if $\sigma-\rho\to 0$ and $\rho\to\delta>0$, we see that $\|Du\|_2\to\infty$. In the second case we have

$$\begin{aligned}
\int_{\Omega(\rho,\sigma)} |Du|^2\,dx &= \int_{\Omega(\rho,\sigma)} K(|x|)F(u)\,dx\\
&\le \frac{a_1}{2^*S^{2^*}}\sup_{\Omega(\rho,\sigma)} K(|x|)\left[\int_{\Omega(\rho,\sigma)} |Du|^2\,dx\right]^{\frac{2^*}{2}}.
\end{aligned}$$

Since $\sup_{\Omega(\rho,\sigma)} K(|x|) \to 0$, as $\sigma - \rho \to 0$ and $\rho \to 0$, we get

$$\int_{\Omega(\rho,\sigma)} |Du|^2 \, dx \to \infty.$$

(iii) Let $\rho > 0$ and $u \in \mathcal{N}(\rho, \infty)$. By virtue of the Sobolev inequality we have

$$\int_{\Omega(\rho,\infty)} |Du|^2 \, dx \le \frac{a_1}{2^* S^{2^*}} \sup_{\Omega(\rho,\infty)} K(|x|) \left(\int_{\Omega(\rho,\infty)} |Du|^2 \, dx \right)^{\frac{2^*}{2}}.$$

Since $\sup_{\Omega(\rho,\infty)} K(|x|) \to 0$ as $\rho \to \infty$, we must have that $\int_{\Omega(\rho,\infty)} |Du|^2 \, dx \to \infty$ as $\rho \to \infty$ and our claim follows.

(iv) The proof is similar to that of Proposition 4.1 (d) in [3]. □

We now use Proposition 2 and 3 to construct a radial solution u_j^+ of (13) with $u_j^+(0) > 0$ having exactly j nodes $0 < \rho_1^+ < \ldots < \rho_j^+ < \infty$.

We set

$$\varepsilon_i = \begin{cases} + & \text{for } i \text{ even} \\ - & \text{for } i \text{ odd} \end{cases}$$

and define

$$M_+(\rho_1, \ldots, \rho_j) = \sum_{i=0}^{j} M_{\varepsilon_i}(\rho_i, \, \rho_{i+1}).$$

According to Proposition 3, $M_+(\rho_1, \ldots, \rho_j)$ attains its minimum at some point $\rho_1^+ < \ldots < \rho_j^+$. For each $0 \le i \le j$ and i even, there exists a positive radial solution u_i of problem (14) on $\Omega(\rho_i^+, \, \rho_{i+1}^+)$ such that $\mathcal{F}_+(u_i) = M_+(\rho_i^+, \, \rho_{i+1}^+)$. For each odd i there exists a negative radial solution u_i with $\mathcal{F}_-(u_i) = M_-(\rho_i^+, \, \rho_{i+1}^+)$. We now define $u_j \in D_r^{1,2}(\mathbb{R}_n)$ by

$$u_j(x) = u_i(x) \quad \text{if} \quad x \in \Omega(\rho_i^+, \, \rho_{i+1}^+). \tag{17}$$

To proceed further we need the following global regularity result which corresponds to Lemma 5.1 in [3].

Proposition 4. *The function u_j defined by* (17) *is a solution of* (13) *which has precisely j nodes, that is, $u_j^{-1}(0) = \{\rho_1^+, \ldots, \rho_j^+\}$.*

PROOF: The proof is identical to that of Lemma 5.1 in [3]. Therefore we give only an outline of the proof. For simplicity we denote u_j by u and since u is radially symmetric we also use the notation $u(x) = u(r)$, $r = |x|$. By standard results from the regularity theory for elliptic equations $u \in C^2$ on

$$\mathcal{U} = \{r > 0;\, r \neq \rho_i,\, i = 1, \dots, j\}$$

and satisfies the equation

$$-\big(r^{n-1}u'\big)' = r^{n-1}K(r)f(u). \tag{18}$$

We must show that $u(r)$ satisfies (18) at $r = \rho_i$, $i = 1, \dots, j$. Towards this end it is enough to show that

$$u'_+ = \lim_{r \to \rho_j + 0} u'(r) = \lim_{r \to \rho_j - 0} u'(r) = u'_-.$$

Arguing indirectly we assume that $u'_+ \neq u'_-$ and set $\rho = \rho_{j-1}$, $\sigma = \rho_j$ and $\tau = \rho_{j+1}$. Also, we may assume that $u \geq 0$ on $[\rho, \sigma]$ and $u \leq 0$ and $[\sigma, \tau]$. For a fixed $\delta > 0$ we define a continuous function $v : [\rho, \tau] \to \mathbb{R}$ by

$$v(r) = \begin{cases} u(r) & \text{for } |r - \sigma| \geq \delta, \\ u(\sigma - \delta) + (r - \sigma + \delta)\frac{u(\sigma+\delta) - u(\sigma-\delta)}{2\delta} & \text{for } |r - \sigma| < \delta. \end{cases}$$

Let $\sigma_0 = \sigma_0(\delta) \in (\sigma - \delta, \sigma + \delta)$ be such that $v(\sigma_0) = 0$. By virtue of Lemma 1 there exists $\alpha = \alpha(\delta) > 0$ such that

$$\alpha v \in \mathcal{N}_+(\rho, \sigma_0) \quad \text{and} \quad \beta v \in \mathcal{N}_-(\sigma_0, \tau).$$

We now define a function $w : [\rho, \tau] \to \mathbb{R}$ by

$$w(r) = \begin{cases} \alpha v(r) & \text{for } \rho \leq r \leq \sigma_0, \\ \beta v(r) & \text{for } \sigma_0 \leq r \leq \tau \end{cases}$$

and set

$$\psi(h) = \int_\rho^\tau \big(\frac{1}{2}h'^2 - K(r)F(h)\big)r^{n-1}\,dr.$$

The idea is to show that

$$\psi(w) \leq \psi(u) - \frac{\sigma^{n-1}}{4}\big(u_+{}' - u_-{}'\big)^2\delta + o(\delta). \tag{19}$$

This implies that $\psi(w) < \psi(u)$ for sufficiently small $\delta > 0$. However, this is impossible since $\psi(u) \leq \psi(w)$ for all $w \in \mathcal{N}$. To establish (19) we show, using the convexity of $F(\sqrt{u})$ and $F(-\sqrt{u})$, that

$$\begin{aligned} \psi(w) \leq \psi(u) &+ \left(\int_\rho^{\sigma-\delta} + \int_{\sigma+\delta}^\tau\right)\left[\frac{1}{2}w'^2 - \frac{w^2}{2u}K(r)f(u)\right] r^{n-1}\,dr \\ &+ \int_{\sigma-\delta}^{\sigma+\delta}\left[\frac{1}{2}w'^2 - K(r)F(w) + K(r)F(u)\right] r^{n-1}\,dr \end{aligned} \tag{20}$$

and moreover we need the following asymptotic relations

$$\int_{\sigma-\delta}^{\sigma+\delta} \left(-K(r)F(w) + K(r)F(u)\right) r^{n-1}\, dr = o(\delta), \tag{21}$$

$$\int_{\rho}^{\sigma-\delta} \left(\frac{1}{2}w'^2 - \frac{w}{2u}K(r)f(u)\right) r^{n-1}\, dr = -\frac{\sigma^{n-1}}{2}{u'_-}^2\delta + o(\delta), \tag{22}$$

$$\int_{\sigma+\delta}^{\rho} \left(\frac{1}{2}w'^2 - \frac{w^2}{K}(r)f(u)\right) r^{n-1}\, dr = -\frac{\sigma^{n-1}}{2}{u'_+}^2\delta + o(\delta) \tag{23}$$

and

$$\int_{\delta-\sigma}^{\sigma+\delta} \frac{1}{2}w'^2 r^{n-1}\, dr = \frac{\sigma^{n-1}}{4}\left(u'_+ + u'_-\right)^2\delta + o(\delta). \tag{24}$$

Technical details needed to establish (20)–(24) can be found in [3] (pp. 273–275). Combining (20)–(24) together we get (19) and this completes the proof. □

If we now set

$$\varepsilon_j = \begin{cases} - & \text{for } i \text{ even,} \\ + & \text{for } i \text{ odd} \end{cases}$$

we obtain, by a similar argument, a radial solution u_j^- with $u_j^-(0) < 0$ having precisely j nodes $\rho_1^- < \rho_2^- < \ldots < \rho_j^-$.

The above discussion leads to the following result.

Theorem 2. *For every integer $j \geq 0$ there exist radial solutions u_j^+ and u_j^- of* (13) *with $u_j^-(0) < 0 < u_j^+(0)$ and having exactly j nodes $0 \leq \rho_1^\pm < \rho_2^\pm < \ldots < \rho_j^\pm$.*

We close this paper with a description of a decay at infinity of solutions of (13) obtained in Theorem 2. Since these solutions are radially symmetric we have $|u(x)| = O\big(|x|^{\frac{2-n}{2}}\big)$ as $|x| \to \infty$ (see [8]). It is known that positive solutions of (13) (see [7]), under suitable assumptions on a nonlinearity f, behave like $O\big(|x|^{2-n}\big)$ as $|x| \to \infty$. For nodal solutions we can give the following result.

Proposition 5. *Let u be a nodal solution of* (13) *constructed in Theorem* 2. *Suppose that $K(|x|) = o(|x|^{-\frac{n+2}{2}})$ as $|x| \to \infty$. Then $|u(x)| = O(|x|^{2-n})$.*

Proof: It is known (see [10]) that the function

$$U(x) = C(n)\big(1 + |x|^2\big)^{-\frac{n-2}{2}},\ x \in \mathbb{R}$$

satisfies the equation

$$-\Delta u = u^{2^*-1} \quad \text{in} \quad \mathbb{R}_n,$$

where $C(n) > 0$ is a constant depending on n. To establish the asymptotic behaviour of u as $|x| \to \infty$ we compare u with U for large $|x|$. We only consider the case $u > 0$ for $|x| > \rho_j$. Since u is radially symmetric we have

$$u(x) \leq C_1 |x|^{-\frac{n-2}{2}} \quad \text{for} \quad |x| > \rho_j,$$

where $C_1 > 0$ is a constant. We set

$$A(R) = \frac{C_1}{C(n)} R^{-\frac{n-2}{2}} (1 + R^2)^{\frac{n-2}{2}} \quad \text{for} \quad R > 0.$$

Choosing $R > \rho_j$ large enough we get

$$K(|x|) C_1^{2^*-1} a_1 |x|^{-\frac{n+2}{2}} (1 + |x|^2)^{\frac{n+2}{2}} - AC(n) < 0$$

for $|x| \geq R$. Letting $v = u - AU$, we see that $v(x) \leq 0$ for $|x| = R$ and

$$\begin{aligned} -\Delta v &= K(|x|) f(u) - AU^{2^*-1} \\ &\leq (1 + |x|^2)^{-\frac{n+2}{2}} \Big(K(|x|) a_1 C_1^{2^*-1} |x|^{-\frac{n+2}{2}} (1 + |x|^2)^{\frac{n+2}{2}} - AC(n) \Big) \end{aligned}$$

for $|x| \geq R$. We now observe that the right side of this inequality is strictly negative for $|x| \geq R$, with R sufficiently large. Since $\lim_{|x| \to \infty} v(x) = 0$, applying the maximum principle, we deduce that $u(x) \leq AU(x)$ for $|x| \geq R$ and the result follows. □

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