A note on the non-emptiness of the limit of approximate systems

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Abstract. Short proofs of the fact that the limit space of a non-gauged approximate system of non-empty compact uniform spaces is non-empty and of two related results are given.

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An *approximate inverse system* (AIS) of uniform spaces $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A)$ consists of a directed set A with respect to a transitive and anti-reflexive relation $<$, a uniform space $(X_\alpha, \mathcal{U}_\alpha)$ for each α in A and, for $\alpha < \beta$, a uniformly continuous function $p_{\alpha\beta} : X_\beta \to X_\alpha$ satisfying the condition

(AIS) For each α in A and U in $\mathcal{U}_\alpha$, there is α' in A such that $\alpha < \alpha'$ and for $\alpha' < \beta < \gamma$, $|p_{\alpha\beta}\, p_{\beta\gamma} - p_{\alpha\gamma}| < U$, i.e. $(p_{\alpha\beta}\, p_{\beta\gamma}(x), p_{\alpha\gamma}(x)) \in U$ for each x in X_γ.

Here uniform spaces are not necessarily Hausdorff and entourages are taken to be symmetric. The definition of approximate systems just given was first considered in [1] and simplifies the original definition of approximate systems of compacta introduced by Mardešić and Rubin [3]. Their approximate systems satisfy two additional conditions, (A1) and (A3), and Mardešić in more recent papers such as [2] calls such systems *gauged approximate systems*.

In the sequel, we consider a fixed AIS $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A)$. Its limit space X is the subspace of the product $\prod(X_\alpha : \alpha \in A)$ consisting of all points $x = (x_\alpha)$ such that for each α in A, x_α is the limit of the net $\{p_{\alpha\beta}(x_\beta) : \alpha < \beta\}$. This means that for each U in $\mathcal{U}_\alpha$, there is α' such that $\alpha < \alpha'$ and for $\alpha' < \beta$, $|p_{\alpha\beta}(x_\beta) - x_\alpha| < U$. Here U can be taken to be open or even closed in $X_\alpha \times X_\alpha$ as such entourages form a base of $\mathcal{U}_\alpha$. The restriction to X of the canonical projection from the product to X_α will be denoted by p_α. The purpose of this note is to give short proofs of the following results in their most general formulation, correcting thus the impression created by the review 93h:54009 of [1] in Mathematical Reviews, which contains the statement that "all these generalizations lead to situations ... with empty limits".

Theorem 1. *In an AIS* $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A)$ *consisting of compact spaces, consider an open set* G *of some* X_{α^*} *containing* $p_{\alpha^*}(X)$*. Then there is* α' *in* A *such that* $\alpha^* < \alpha'$ *and for* $\alpha' < \beta$, $p_{\alpha^*\beta}(X_\beta) \subset G$.

Corollary 1. *If each* X_α *is compact and each* $p_{\alpha\beta}$ *is surjective, then each* p_α *is surjective.*

Corollary 2. *If each* X_α *is compact and non-empty, then so is* X.

Corollary 2 for gauged approximate systems of metric compacta appeared first in [3, Theorem 1], and for gauged approximate systems of compact Hausdorff spaces in [5, Theorem 4.1]. Theorem 1 and Corollary 1 for gauged approximate systems of metric compacta are proved in [4, Theorem 1 and Corollary 1], assuming Corollary 2. In all cases, the given proofs are lengthy and they appeal to both axioms (A1) and (A3). Finally, Mardešić [2, Theorem 6] derives Corollary 2 (as well as several other results) for Hausdorff spaces from a result that to each AIS of such spaces assigns a gauged AIS consisting of the same spaces and having the same limit space. As is well known, the inverse limit of non-empty, compact and Hausdorff spaces is not empty, but none of the assumptions on the spaces can be dropped.

Example 1. Let $X_n = \{n, n+1, n+2, \dots\}$ with uniformity consisting only of $X_n \times X_n$ for each n in N and, for $m < n$, let p_{mn} denote the inclusion of X_n in X_m. Then (X_n, p_{mn}, N) is an inverse limit system with empty limit while its limit space as an AIS is $\prod(X_n : n \in N)$.

The proof of Theorem 1 relies on the following result.

Lemma 1. *Let* Y, Z *be uniform spaces,* U *a closed entourage of* Z *and* $f, g : Y \to Z$ *continuous functions. Then* $F = \{x \in Y : |f(x) - g(x)| < U\}$ *is a closed subset of* Y.

PROOF: If $x \notin F$, since U is closed in $Z \times Z$, there is an entourage V of Z such that $B(f(x), V) \times B(g(x), V) \cap U = \emptyset$, where $B(y, V)$ denotes the set $\{z \in Z : |y - z| < V\}$. But then the neighbourhood $f^{-1}(B(f(x), V)) \cap g^{-1}(B(g(x), V))$ of x is disjoint from F. Hence F is closed. □

Proof of Theorem 1. Assume that $B = \{\beta \in A : p_{\alpha^*\beta}(X_\beta) \not\subset G\}$ is cofinal in A. Let M consist of all triples (α, α', U) such that $\alpha < \alpha'$, U is a closed member of $\mathcal{U}_\alpha$ and for $\alpha' \leq \beta < \gamma$, $|p_{\alpha\beta} p_{\beta\gamma} - p_{\alpha\gamma}| < U$. Note that if (α, α', U) is in M, then so is (α, β, U) whenever $\alpha' < \beta$. For each $\mu = (\alpha, \alpha', U)$ in M, define

$$F_\mu = \Big\{x = (x_\alpha) \in \prod X_\alpha : |p_{\alpha\alpha'}(x_{a'}) - x_\alpha| < U \ \text{ and } \ x_{\alpha^*} \notin G\Big\}.$$

Since each p_α is continuous, it follows from Lemma 1 that each F_μ is closed in the product. Consider next a finite subset L of M. Then there is by assumption an element β of B that is greater than α^* and the second coordinate of every member of L, and a point b of X_β such that $p_{\alpha^*\beta}(b) \notin G$. The cofinality of B implies that each space of our AIS is non-empty, so that there is a member $x = (x_\alpha)$ of the product such that $x_\beta = b$ and, for $\alpha < \beta$, $x_\alpha = p_{\alpha\beta}(b)$. Now for each $\lambda = (\alpha, \alpha', U)$ in L, as $\alpha < \alpha' < \beta$, we have $|p_{\alpha\alpha'}(x_{\alpha'}) - x_\alpha| = |p_{\alpha\alpha'}\, p_{\alpha'\beta}(b) - p_{\alpha\beta}(b)| < U$. Since evidently $x_{\alpha^*} \notin G$, then x belongs to F_λ for each λ in L. Thus, the closed family $\{F_\mu : \mu \in M\}$ of the compact $\prod X_\alpha$ has the finite intersection property. Hence there is a point $y = (y_\alpha)$ of the product that belongs to each F_μ. Evidently, $p_{\alpha^*}(y) \notin G$ and to complete the proof it suffices to show $y \in X$. By (AIS), for each α in A and closed U in $\mathcal{U}_\alpha$, there is α' such that $(\alpha, \alpha', U) \in M$. Therefore, for $\alpha' < \beta$, $\mu = (\alpha, \beta, U) \in M$ so that $y \in F_\mu$ and hence $|p_{\alpha\beta}(y_\beta) - y_\alpha| < U$. This shows that $y \in X$ and completes the proof. □

Proof of Corollary 1. If a, b have the same closure in X_α, then for all U in $\mathcal{U}_\alpha$, $|a - b| < U$, and a net converges to a iff it converges to b. Consequently, if $x = (x_\alpha)$ is in X with $x_\alpha = a$, $y_\alpha = b$ and, for $\alpha \neq \beta$, $y_\beta = x_\beta$, then $y = (y_\alpha) \in X$. Thus, if a is not in $p_\alpha(X)$ and G is the complement of the closure of a, then $p_\alpha(X) \subset G$. By Theorem 1, $p_{\alpha\beta}(X_\beta) \subset G$ for eventually all β, contradicting the assumption that $p_{\alpha\beta}$ is surjective. □

Proof of Corollary 2. As a closed subspace of the product, X is compact. If $X = \emptyset$, for any α in A, $p_{\alpha\beta}(X_\beta) = \emptyset$ and hence $X_\beta = \emptyset$ for eventually all β. □

Corrections. In conclusion, we take the opportunity to note some minor corrections to our paper [1]. In Lemma 3, the map f need not be assumed to be locally finite, and $h(x)$ lies in the carrier of $f(x)$. In Lemma 4, the maps f_i need not be assumed locally finite. In Propositions 11 and 12, the bonding maps should not be claimed to be surjective.

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