On the value distribution of a class of arithmetic functions

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Abstract. This article deals with the value distribution of multiplicative prime-independent arithmetic functions $(\alpha(n))$ with $\alpha(n) = 1$ if n is N-free ($N \geq 2$ a fixed integer), $\alpha(n) > 1$ else, and $\alpha(2^n) \to \infty$. An asymptotic result is established with an error term probably definitive on the basis of the present knowledge about the zeros of the zeta-function. Applications to the enumerative functions of Abelian groups and of semisimple rings of given finite order are discussed.

Keywords: arithmetic functions, value distribution, finite Abelian groups

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1. Introduction and statement of result

Let $(\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence of real numbers such that, for a certain integer $N \geq 2$,

$$\begin{aligned} \lambda_n &= 1 \quad \text{for } n = 0, \dots, N-1, \\ \lambda_n &> 1 \quad \text{for } n \geq N, \\ \lambda_n &\to \infty \quad \text{for } n \to \infty\,. \end{aligned} \tag{1.1}$$

We define an arithmetic function $(\alpha(n))_{n\in\mathbb{N}}$ in the following way: If

$$n = \prod_{p\in\mathbb{P}} p^{\eta(p)}$$

is the canonical prime factor decomposition of the positive integer n, then

$$\alpha(n) := \prod_{p\in\mathbb{P}} \lambda_{\eta(p)}\,.$$

Obviously, $(\alpha(n))_{n\in\mathbb{N}}$ is multiplicative and prime independent. Let $\mathcal{Y}$ denote the set of values attained by $\alpha(n)$; under multiplication, this is a semigroup containing 1. In the present paper, we shall be concerned with the value distribution of such sequences $\alpha(n)$, i.e., we will derive an asymptotic formula for the quantity

$$A_y(x) = \#\{n \in \mathbb{N}, n \leq x : \ \alpha(n) = y\},$$

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where y is some fixed element of $\mathcal{Y}$ and x is a large real variable.

The classic example for the problem under consideration is of course the counting function of (isomorphism classes of) Abelian groups of order n. Here $N = 2$, and $\lambda_n = P(n)$, the number of unrestricted partitions of n elements. In this case, the behaviour of $A_y(x)$ has been studied by Ivić [2], Krätzel [4], [6], and Krätzel and Wolke [7].

The objective of this article is to derive an asymptotic formula for $A_y(x)$ which will be both sharper and more general than those contained in the papers cited. In fact, our result can be called "definitive" on the basis of the present knowledge concerning zero-free regions of the Riemann zeta-function; the order term obtained corresponds to the hitherto sharpest one in the Prime Number Theorem.

Theorem. *For any $y \in \mathcal{Y}$ fixed, define $d_* \in \mathbb{N}_0$ as the maximal integer for which $\frac{y}{\lambda_N^{d_*}} \in \mathcal{Y}$. Then there holds an asymptotic formula*

$$A_y(x) = C_y x + x^{1/N} \sum_{l=2}^{M(x)} \frac{\mathcal{P}_l(\log\log x)}{(\log x)^l} + O(x^{1/N}\delta_0(x)).$$

For $d_* = 0$ (i.e., $\frac{y}{\lambda_N} \notin \mathcal{Y}$),

$$A_y(x) = C_y x + O(x^{1/N}\delta_0(x)).$$

Here $\mathcal{P}_l(\cdot)$ are polynomials of degree $\le d_* - 1$, their coefficients are computable real numbers $\ll (b_* l)^l$, with appropriate $b_* > 0$. C_y is a computable constant, and

$$M(x) := [c^*(\log x)^{3/5}(\log\log x)^{-6/5}]. \tag{1.2}$$

Finally, here and throughout,

$$\delta_j(u) := \exp\left(-c_j(\log(2+u))^{3/5}(\log\log(2+u))^{-1/5}\right)$$

for $u \ge 1$ and suitable positive constants c_j, $j = 0, 1, \dots$. All coefficients and constants may depend on the sequence (λ_n) and on the element $y \in \mathcal{Y}$.

2. Preparations for the proof

Throughout the paper, b and c (also with a subscript or superscript) denote positive constants which may depend on the sequence (λ_n) and on the element $y \in \mathcal{Y}$. (This applies to all O- and $\ll$-constants as well.)

Let G be any analytic function without zeros on a certain simply connected domain S of $\mathbb{C}$ which contains the real line to the right of $s = \sigma_0$, where $\sigma_0 \in \mathbb{R}_0^+$. Suppose that $G(s) \in \mathbb{R}^+$ for real $s > \sigma_0$, then we define, for $s \in S$,

$$\log G(s) = \log G(1+\sigma_0) + \int_{1+\sigma_0}^{s} \frac{G'(z)}{G(z)}\,\mathrm{d}z,$$

the path of integration being completely contained in S but otherwise arbitrary.

In our analysis, S will usually be a domain symmetric with respect to the real line, with a "cut" along $L = \{s \in \mathbb{R} : s \le \sigma_0\}$ (such that $S \cap L = \emptyset$). We will join in the common abuse of terminology to speak of an "upper" and "lower edge" of $L \cap \partial S$, on which $\log G(s)$ are attributed two different values, depending on whether L is approached from above or from below.

By a *decomposition* $\mathbf{I}$ of some fixed $y \in \mathcal{Y}$, we mean a finite nondecreasing sequence $\mathbf{I} = (i(j))_{j=1}^{J}$ of integers $i(j) \ge N$ with the property

$$y = \prod_{j=1}^{J} \lambda_{i(j)} \,.$$

Clearly, only finitely many decompositions $\mathbf{I}$ correspond to each $y \in \mathcal{Y}$. By $\mathbb{N}_{\mathbf{I}}$ we denote the set of all natural numbers n of the form

$$n = n_1 \prod_{j=1}^{J} p_j^{i(j)} \tag{2.1}$$

where $p_1, \dots, p_J$ are distinct primes and n_1 is an N-free natural number which is relative prime to $p_1 \dots p_J$. Thus $n \in \mathbb{N}_{\mathbf{I}}$ always implies that $\alpha(n) = y$.

Finally, we shall denote by $\mathbb{H}$ the set of all complex-valued functions defined by an absolutely convergent (ordinary) Dirichlet series on the half-plane $\operatorname{Re} s > \frac{1}{N+1}$; elements of $\mathbb{H}$ will be denoted by h, usually with some subscript and/or superscript.

We are now ready to formulate our first auxiliary result.

Lemma 1 (cf. Krätzel and Wolke [7, Lemma 1]). *For arbitrary fixed $y \in \mathcal{Y}$ and a fixed decomposition $\mathbf{I}$ of y, define*

$$\phi_{\mathbf{I}}(s) = \sum_{n \in \mathbb{N}_{\mathbf{I}}} \frac{1}{n^s} \quad \text{for} \quad \operatorname{Re} s > 1 \,.$$

Then there exists a representation

$$\phi_{\mathbf{I}}(s) = \frac{\zeta(s)}{\zeta(Ns)} \sum_{k=0}^{d} h_k(s) (\log \zeta(Ns))^k$$

with $h_k \in \mathbb{H}$ and $d \in \mathbb{N}$ maximal such that $i(d) = N$ ($d = 0$ if $i(1) > N$ or if $y = 1$). This provides an analytic continuation of the function $\phi_{\mathbf{I}}$ to any simply connected zero-free region of $\zeta(Ns)$ contained in $\operatorname{Re} s > \frac{1}{N+1}$ which avoids the "cut" $\{s \in \mathbb{R} :\ s \le \frac{1}{N}\}$.

Proof: For $\operatorname{Re} s > 1$ (throughout the sequel), we may write

$$\phi_{\mathbf{I}}(s) = \sum_{p_1,\dots,p_J}{}' \left\{ \left(\prod_{j=1}^{J} p_j^{i(j)} \right)^{-s} \sum_{\substack{n_1 \ N\text{-free} \\ (n_1,P)=1}} n_1^{-s} \right\},$$

where $\sum'$ means summation over all J-tuples of *distinct* primes $p_1,\dots,p_J$, and $P = p_1 \dots p_J$ for short. For $p_1,\dots,p_J$ fixed, let $\chi_{N,P}$ denote the characteristic function of the positive integers n which are N-free and relative prime to P. Then the inner sum is

$$\sum_{n\in\mathbb{N}} \chi_{N,P}(n) n^{-s} = \prod_{p\in\mathbb{P}} \left(\sum_{k=0}^{\infty} \chi_{N,P}(p^k) p^{-ks} \right) =$$
$$= \prod_{p\nmid P} \left(\sum_{k=0}^{N-1} p^{-ks} \right) = \frac{\zeta(s)}{\zeta(Ns)} \prod_{p\mid P} \frac{1-p^{-s}}{1-p^{-Ns}}\,.$$

Therefore,

$$\phi_{\mathbf{I}}(s) = \frac{\zeta(s)}{\zeta(Ns)} \sum_{p_1,\dots,p_J}{}' \prod_{j=1}^{J} \left(p_j^{-i(j)s} \frac{1-p_j^{-s}}{1-p_j^{-Ns}} \right). \tag{2.2}$$

We split up this sum in the form

$$\sum_{p_1,\dots,p_d}{}' \prod_{j=1}^{d} p_j^{-Ns} \frac{1-p_j^{-s}}{1-p_j^{-Ns}} \left(\sum_{\substack{p_{d+1},\dots,p_J: \\ (P',P'')=1}}{}' \prod_{j=d+1}^{J} p_j^{-i(j)s} \frac{1-p_j^{-s}}{1-p_j^{-Ns}} \right), \tag{$*$}$$

where $P' = p_1 \dots p_d$, $P'' = p_{d+1} \dots p_J$. With $p_1,\dots,p_d$ fixed, we apply Vinogradov's lemma to the inner sum (μ denotes the Möbius function):

$$\sum_{\substack{p_{d+1},\dots,p_J: \\ (P',P'')=1}}{}' \prod_{j=d+1}^{J} p_j^{-i(j)s} \frac{1-p_j^{-s}}{1-p_j^{-Ns}} =$$
$$= \sum_{t} \mu(t) \left(\sum_{\substack{p_{d+1},\dots,p_J: \\ t\mid(P',P'')}}{}' \prod_{j=d+1}^{J} p_j^{-i(j)s} \frac{1-p_j^{-s}}{1-p_j^{-Ns}} \right).$$

For $t=1$, the large bracket represents a function of the class $\mathbb{H}$. For $t>1$, some of $p_{d+1},\dots,p_J$ coincide with some of $p_1,\dots,p_d$. After resubstitution into $(*)$, these primes have exponents $-N's$ with $N' > N$, thus they contribute a factor which belongs to $\mathbb{H}$, too. Repeated application of this trick finally yields

$$\phi_{\mathbf{I}}(s) = \frac{\zeta(s)}{\zeta(Ns)} \sum_{k=0}^{d} h_k^*(s) \sum_{p_1,\dots,p_k}{}' \prod_{j=1}^{k} p_j^{-Ns} \frac{1-p_j^{-s}}{1-p_j^{-Ns}} ,$$

with $h_k^* \in \mathbb{H}$. By the inclusion-exclusion principle, we can get rid of the condition that the primes be distinct. Using the well-known identity

$$\sum_{p\in\mathbb{P}} p^{-Ns} \frac{1-p^{-s}}{1-p^{-Ns}} = \sum_{p\in\mathbb{P}} p^{-Ns} + h^{**}(s) = \log \zeta(Ns) + h^{***}(s),$$

with $h^{**}, h^{***} \in \mathbb{H}$, we complete the proof of Lemma 1. □

One important ingredient in our argument — in order to obtain the sharp error term we have stated — will be some information about the coefficients of a Dirichlet series arising from the above factorization of $\phi_{\mathbf{I}}(s)$.

Lemma 2. *For fixed $y \in \mathcal{Y}$ and a fixed decomposition $\mathbf{I}$ of y, let the arithmetic function $\beta(n)$ be defined by the Dirichlet series representation*

$$Z(s) := \frac{\phi_{\mathbf{I}}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} \qquad (\operatorname{Re} s > 1)\,.$$

Then it follows that

(i) *$\beta(n) \neq 0$ implies that n is N-full*,*
(ii) *$\beta(n) \ll (\log n)^J$ for all $n \geq 2$.*

Proof: Recalling (2.2), we have (at least for $\operatorname{Re} s > 1$)

$$\sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} = \prod_{p\in\mathbb{P}} \left(1-p^{-Ns}\right) \sum_{p_1,\dots,p_J}{}' \prod_{j=1}^{J} \left(p_j^{-i(j)s} \frac{1-p_j^{-s}}{1-p_j^{-Ns}} \right) .$$

Expanding all the products on the right hand side obviously gives a series over n^{-s} where n ranges only over the N-full integers: This shows the first assertion.

To prove (ii), consider an arbitrary $n \in \mathbb{N}$ with $\beta(n) \neq 0$; necessarily n is N-full. If $\chi_{\mathbf{I}}$ denotes the characteristic function of $\mathbb{N}_{\mathbf{I}}$ it is clear from the definition that

$$\beta(n) = \sum_{ml=n} \mu(m)\chi_{\mathbf{I}}(l)\,.$$

*A positive integer n is called N-full if in its canonical factorization into prime powers all nonzero exponents are $\geq N$.

Therefore, it suffices to estimate the cardinality of the set

$$\mathcal{N} = \{(m,l) \in \mathbb{N}^2 : \ ml = n, \ \mu(m)\chi_{\mathbf{I}}(l) \neq 0\,\} .$$

As usual, let

$$\omega_K(k) = \sum_{p\in\mathbb{P}:\ p^K|k} 1, \qquad \omega(k) = \omega_1(k) .$$

If $(m,l) \in \mathcal{N}$, it follows from (2.1) that $\omega_N(l) = J$. Because n is N-full, m must contain at least $\omega(n) - J$ of the prime factors of n. Since m is squarefree, the number of possible values for m is

$$\leq \sum_{k=\omega(n)-J}^{\omega(n)} \binom{\omega(n)}{k} = \sum_{k=0}^{J} \binom{\omega(n)}{k} \ll (\omega(n))^J \ll (\log n)^J .$$

□

In our next Lemma, we summarize the present state of art about zero-free regions of the Riemann zeta-function.

Lemma 3 (see Ivić [3]). *Define for short*

$$\psi(t) = (\log t)^{2/3}(\log\log t)^{1/3} \qquad (t \geq 3)$$

and, for positive constants $b_1 \geq 3$ *and* b_2,

$$\eta(t) = \begin{cases} 1 - b_0 := 1 - \frac{b_2}{\psi(b_1)} & \text{for } |t| \leq b_1, \\ 1 - \frac{b_2}{\psi(|t|)} & \text{for } |t| \geq b_1 . \end{cases}$$

Then there exist values of b_1, b_2, b_3 *such that for all* $s = \sigma + it$ *with*

$$\sigma \geq \eta(t)$$

it is true that

$$\zeta(s) \neq 0,$$

and

$$(\zeta(s))^{-1} \ll (\log(2+|t|))^{b_3} ,$$
$$\log\zeta(s) \ll (\log(2+|t|))^{b_3} .$$

This deep result enables us to derive an asymptotic formula for the Dirichlet summatory function of $\beta(n)$.

Lemma 4. *For* $u \to \infty$,

$$B(u) := \sum_{n \le u} \beta(n) = I(u) + R(u)$$

where

$$I(u) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} Z(\frac{s}{N}) u^{s/N} \, \frac{\mathrm{d}s}{s} \,,$$

and

$$R(u) \ll u^{1/N} \delta_1(u)$$

for some $c_1 > 0$. *Here* $\mathcal{C}_0$ *is the circle* $|s-1| = b_0$ (b_0 *from Lemma* 3), *with positive orientation, starting and ending at* $1 - b_0$.

Proof: By a version of Perron's formula,

$$B_1(u) := \int_1^u B(w^N) \, \mathrm{d}w = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Z(\frac{s}{N}) \, \frac{u^{s+1}}{s(s+1)} \, \mathrm{d}s \,.$$

Now let $\mathcal{C}_1$ denote the path from $1 - i\infty$ to $1 - b_0$, $\mathcal{C}_2$ the path from $1 - b_0$ to $1 + i\infty$, both along $\sigma = \eta(t)$ ($s = \sigma + it$ as usual), and put $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_0 \cup \mathcal{C}_2$. ($b_0$ and $\eta(t)$ are defined as in Lemma 3.) In view of Lemmas 1 and 3,

$$Z(\frac{s}{N}) \ll (\log(2 + |t|))^{b_4} \qquad \text{for} \;\; \sigma \ge \eta(t) \,.$$

Thus we can replace the line of integration in the above integral by $\mathcal{C}$. Defining

$$T = \frac{1}{\delta_2(u)}$$

(with suitable $c_2 > 0$), we see that (for $j = 1, 2$)

$$\int_{\mathcal{C}_j} Z(\frac{s}{N}) \frac{u^{s+1}}{s(s+1)} \, \mathrm{d}s = \int_{|t| \ge T} + \int_{|t| \le T} \ll$$
$$\ll \frac{u^2}{T} (\log T)^{b_4} + u^{1+\eta(T)} \ll u^2 \delta_3(u),$$

hence

(2.3) $$B_1(u) = I_1(u) + O(u^2 \delta_3(u)),$$

where

(2.4) $$I_1(u) := \frac{1}{2\pi i} \int_{\mathcal{C}_0} Z(\frac{s}{N}) \frac{u^{s+1}}{s(s+1)} \, \mathrm{d}s \,.$$

Employing a technique due to Rieger [11], we now put, for $w \geq 1$,

$$(w) := B(w^N) - I(w^N) + I(1),$$

then (2.3) implies that

$$\int_1^u f(w)\ \mathrm{d}w \ll u^2 \delta_3(u)\,. \tag{2.5}$$

We have to estimate the difference $f(w_1) - f(w_2)$ for $w_1 > w_2$. If $Q_N(v)$ denotes the number of N-full integers $\leq v$, we deduce from Lemma 2 that

$$\begin{aligned} |B(w_1^N) - B(w_2^N)| &\ll (\log w_1)^J \left(Q_N(w_1^N) - Q_N(w_2^N)\right) \ll \\ &\ll (w_1 - w_2)(\log w_1)^J + w_1^\theta \qquad (\theta < 1), \end{aligned} \tag{2.6}$$

where the last estimate is an immediate consequence of the asymptotique formula for $Q_N(v)$. (See Krätzel [5, p. 280].)

On the other hand,

$$I(w_1^N) - I(w_2^N) = \int_{w_2}^{w_1} \left(\frac{1}{2\pi i} \int_{\mathcal{C}_0} Z(\frac{s}{N}) u^{s-1}\ \mathrm{d}s \right) \mathrm{d}u \ll w_1 - w_2\,. \tag{2.7}$$

To see this, we replace $\mathcal{C}_0$ by $\mathcal{C}_0^*(u)$ which we define as the boundary of

$$\{s \in \mathbb{C}:\ |s-1| \leq b_0,\ \mathrm{Re}\, s \leq 1 + \frac{1}{\log(2u)}\}, \tag{2.8}$$

(with positive orientation, starting and ending at $1 - b_0$), and observe that

$$|Z(\frac{s}{N})| \ll \frac{|\log \zeta(s)|^d}{|\zeta(s)|} \ll |\log|s-1||^d\ |s-1| \ll 1 \tag{2.9}$$

for s close to 1, $\mathrm{Re}\, s \geq 1$.

Now (2.5), (2.6), and (2.7) establish just the requirements of "Hilfssatz 2" in Rieger [11]. The latter implies that

$$B(w^N) = I(w^N) + O(w\delta_4(w))\,.$$

Putting $u = w^N$, we complete the proof of Lemma 4. □

We conclude this section with a simple but useful estimate for integrals related to the incomplete Gamma function.

Lemma 5. *For $\tau, k \in \mathbb{R}$, $W \geq 3$, suppose that*

$$\frac{\max(\tau, k)}{W} \leq \frac{1}{4}\,. \tag{2.10}$$

Then

$$\int_W^\infty u^\tau e^{-u} (\log u)^k \,\mathrm{d}u \leq 2 W^\tau e^{-W} (\log W)^k\,.$$

Proof: Following an idea of De Koninck and Ivić [1, p. 12], we put

$$g(u) = \log\left(u^\tau e^{-u} (\log u)^k\right),$$

then

$$g'(u) = -1 + \frac{\tau}{u} + \frac{k}{u}\frac{1}{\log u} \leq -\frac{1}{2} \quad \text{for} \ \ u \geq W.$$

Hence, by the mean value theorem,

$$g(u) \leq g(W) - \tfrac{1}{2}(u - W) \quad \text{for} \ \ u \geq W.$$

Therefore, the integral in question is

$$\int_W^\infty \exp(g(u))\,\mathrm{d}u \leq \exp(g(W) + \tfrac{1}{2}W) \int_W^\infty e^{-u/2} \,\mathrm{d}u = 2W^\tau e^{-W} (\log W)^k\,.$$

□

3. Proof of the Theorem

We employ an argument which is sometimes called the Selberg-Deligne method and originates in a paper of Selberg [12]. An enlightening exposition can be found in the textbook of De Koninck and Ivić [1].

A salient point of our approach is that we do not evaluate $\sum \chi_{\mathbf{I}}(n)$ directly by Perron's formula but start from an elementary convolution argument based on the identity

$$\chi_{\mathbf{I}}(n) = \sum_{m|n} \beta(m), \tag{3.1}$$

along with the result of Lemma 4 on $\sum \beta(m)$. We define

$$z = z(x) = x\delta_5(x), \qquad q = q(x) = \frac{x}{z} = \frac{1}{\delta_5(x)}, \tag{3.2}$$

with a positive constant c_5 remaining at our disposition. By (3.1),

$$\sum_{n \leq x} \chi_{\mathbf{I}}(n) = \sum_{m \leq z} \beta(m)[\frac{x}{m}] \ + \sum_{k \leq q} B(\frac{x}{k}) \quad - \ B(z)[q]\,. \tag{3.3}$$

We are going to evaluate the different terms occurring in this formula. Writing $\{\cdot\}$ for the fractional part, we first note that, by Lemma 2,

$$\sum_{m\le z}\beta(m)\{\frac{x}{m}\}\ll(\log z)^J Q_N(z)\ll z^{1/N}(\log x)^J\,. \tag{3.4}$$

Furthermore, in view of Lemma 4,

$$\begin{aligned}\sum_{m>z}\frac{\beta(m)}{m}&=\int_{z+}^{\infty}\frac{1}{u}\,\mathrm{d}B(u)=\\&=\int_{z}^{\infty}\frac{1}{u}\,I'(u)\,\mathrm{d}u+\int_{z+}^{\infty}\frac{1}{u}\,\mathrm{d}R(u)=\\&=\int_{z}^{\infty}\frac{1}{u}\,I'(u)\,\mathrm{d}u\;+\;O(z^{-1+1/N}\delta_1(z))\,.\end{aligned} \tag{3.5}$$

Thus we obtain

$$\sum_{m\le z}\beta(m)[\frac{x}{m}]=A_{\mathbf{I}}x-x\int_{z}^{\infty}\frac{1}{u}\,I'(u)\,\mathrm{d}u\;+\;O(x^{1/N}\delta_6(x)), \tag{3.6}$$

with

$$A_{\mathbf{I}}=\sum_{m=1}^{\infty}\frac{\beta(m)}{m}\,,$$

by a suitable choice of c_5 and c_6. On the other hand,

$$\sum_{k\le q}R(\frac{x}{k})\ll x^{1/N}\sum_{k\le q}k^{-1/N}\delta_1(\frac{x}{k})\ll x^{1/N}\,q^{1-1/N}\;\delta_1(z)\ll x^{1/N}\,\delta_7(x), \tag{3.7}$$

and

$$\begin{aligned}\sum_{k\le q}I(\frac{x}{k})&=\int_{\frac12}^{q}I(\frac{x}{u})\,\mathrm{d}[u]=\\&=I(\frac{x}{q})[q]\;+\;x\int_1^q\frac{[u]}{u^2}I'(\frac{x}{u})\,\mathrm{d}u=\\&=I(z)[q]\;+\;x\int_z^x I'(v)\,\frac{\mathrm{d}v}{v}\;-\;x\int_1^q I'(\frac{x}{u})\frac{\{u\}}{u^2}\,\mathrm{d}u,\end{aligned}$$

by the substitution $v=\dfrac{x}{u}$ in the next-to-last integral. Using this together with (3.7) and (3.6) in (3.3), we arrive at

$$\begin{aligned}\sum_{n\le x}\chi_{\mathbf{I}}(n)&=\\&=A_{\mathbf{I}}x\;-\;x\int_x^{\infty}I'(u)\,\frac{\mathrm{d}u}{u}\;-\;x\int_1^q I'(\frac{x}{u})\frac{\{u\}}{u^2}\,\mathrm{d}u\;+\;O(x^{1/N}\delta_8(x)),\end{aligned} \tag{3.8}$$

where

$$I'(u) = \frac{1}{2\pi i} \int_{\frac{1}{N}\mathcal{C}_0} Z(s) u^{s-1} \, \mathrm{d}s \,. \tag{3.9}$$

To evaluate the two remaining integrals, we define

$$S(w,s) := \int_w^\infty \{u\} \, u^{-s-1} \, \mathrm{d}u$$

and

$$T(x,w) := \frac{1}{2\pi i} \int_{\frac{1}{N}\mathcal{C}_0} Z(s) S(w,s) x^s \, \mathrm{d}s,$$

for positive reals w and x and complex s with $\operatorname{Re} s > 0$. Interchanging the order of integration, we see from (3.9) that

$$T(x,w) = x \int_w^\infty \frac{\{u\}}{u^2} I'(\frac{x}{u}) \, \mathrm{d}u \,.$$

Consequently, we obtain for the last integral in (3.8)

$$-x \int_1^q I'(\frac{x}{u}) \frac{\{u\}}{u^2} \, \mathrm{d}u = -x \int_1^\infty I'(\frac{x}{u}) \frac{\{u\}}{u^2} \, \mathrm{d}u \; + \; T(x,q)$$

or, in view of the well-known identity

$$\int_1^\infty \{u\} \, u^{-s-1} \, \mathrm{d}u = \frac{1}{s-1} - \frac{\zeta(s)}{s}$$

(valid for Re $s > 0$),

$$-x \int_1^q I'(\frac{x}{u}) \frac{\{u\}}{u^2} \, \mathrm{d}u = \frac{1}{2\pi i} \int_{\frac{1}{N}\mathcal{C}_0} Z(s) \left(\zeta(s) - \frac{s}{s-1} \right) x^s \, \frac{\mathrm{d}s}{s} \; + \; T(x,q) \,. \tag{3.10}$$

Similarly,

$$\int_x^\infty I'(u) \frac{\mathrm{d}u}{u} = \int_x^\infty \left(\frac{1}{2\pi i} \int_{\frac{1}{N}\mathcal{C}_0} Z(s) u^{s-1} \, \mathrm{d}s \right) \frac{\mathrm{d}u}{u} =$$

$$= \frac{1}{2\pi i} \int_{\frac{1}{N}\mathcal{C}_0} Z(s) \left(\int_x^\infty u^{s-2} \, \mathrm{d}u \right) \, \mathrm{d}s = -\frac{1}{2\pi i} \int_{\frac{1}{N}\mathcal{C}_0} \frac{Z(s)}{s-1} x^{s-1} \, \mathrm{d}s \,.$$

Hence (3.8) may be simplified to

$$\sum_{n \le x} \chi_{\mathbf{I}}(n) = A_{\mathbf{I}} x \; + \; I^*(x) \; + \; T(x,q) \; + \; O(x^{1/N} \delta_8(x)), \tag{3.11}$$

where

$$(3.12)\qquad I^*(x) := \frac{1}{2\pi i}\int_{\frac{1}{N}\mathcal{C}_0} \zeta(s)Z(s)\,x^s\,\frac{\mathrm{d}s}{s} = \frac{1}{2\pi i}\int_{\frac{1}{N}\mathcal{C}_0} \phi_{\mathbf{I}}(s)\,x^s\,\frac{\mathrm{d}s}{s}\,.$$

Our penultimate step is thus to estimate $T(x,q)$. It is clear from the definition that

$$S(w,\sigma+it) \ll w^{-\sigma},$$

hence

$$(3.13)\qquad T(x,q) = -\frac{1}{2\pi i}\int_{\frac{1}{N}\mathcal{C}_0^*(x)} Z(s)S(q,s)\,x^s\,\mathrm{d}s \ll (\frac{x}{q})^{1/N} \ll x^{1/N}\delta_9(x),$$

where $\mathcal{C}_0^*(x)$ is defined as in (2.8). (By (2.9), $Z(s)$ is bounded on $\frac{1}{N}\mathcal{C}_0^*(x)$.) For $d=0$, $\phi_{\mathbf{I}}(s)$ does not contain any logarithmic terms, hence the integrand of $I^*(x)$ is regular in an open disk containing $\mathcal{C}_0$. In this case, $I^*(x)$ vanishes, and the Theorem is immediate from (3.11) and (3.13).

For $d>0$, the proof of the Theorem will be complete if we establish the following last auxiliary result.

Lemma 6. *For a fixed element $y\in\mathcal{Y}$ with $\frac{y}{\lambda_N}\in\mathcal{Y}$ and a fixed decomposition $\mathbf{I}$ of y, the integral $I^*(x)$ defined in (3.12) possesses the asymptotic expansion (as $x\to\infty$)*

$$I^*(x) = x^{1/N}\sum_{l=2}^{M(x)} \frac{\mathcal{P}_{l,\mathbf{I}}(\log\log x)}{(\log x)^l} + O(x^{1/N}\delta_{10}(x))\,.$$

where $M(x)$ is given by (1.2) and $\mathcal{P}_{l,\mathbf{I}}(\cdot)$ are polynomials of degree at most $d-1$ with computable coefficients.

Proof*: We put $x=w^N$ and obtain

$$(3.14)\qquad I^*(w^N) = \frac{1}{2\pi i}\int_{\mathcal{C}_0} \phi_{\mathbf{I}}(\frac{s}{N})w^s\frac{\mathrm{d}s}{s} = \sum_{\kappa=0}^{d}\sum_{k=0}^{\kappa} I^*_{k,\kappa}(w^N),$$

with

$$I^*_{k,\kappa}(w^N) := \frac{1}{2\pi i}\int_{\mathcal{C}_0} H_{k,\kappa}(s)\,(\log(s-1))^k\,w^s\,\mathrm{d}s,$$

where

$$H_{k,\kappa}(s) = (-1)^k\binom{\kappa}{k}\frac{\zeta(\frac{s}{N})}{\zeta(s)}h_\kappa(\frac{s}{N})\,(\log G(s))^{\kappa-k}\,\frac{1}{s}\,,$$

*We give a detailed deduction of this result only for convenience of the reader, without any particular claim of originality. Quite similar arguments may be found, e.g., in Wolke [13], [14], Wu [15], and also in the author's earlier papers [9], [10].

$G(s) = (s-1)\zeta(s)$ and $h_\kappa \in \mathbb{H}$ as in Lemma 1. We note that $H_{k,\kappa}(1) = 0$ and that each $H_{k,\kappa}(s)$ is regular on some open disk around 1 with radius $> b_0$. (We obviously may assume b_0 to be so small that this is true.) With $M = M(w^N)$ according to (1.2), we thus obtain

$$H_{k,\kappa}(s) = \sum_{j=1}^{M-1} \gamma_j (s-1)^j + r_M(s), \tag{3.15}$$

where

$$\gamma_j \ll b^j, \qquad r_M(s) \ll b^M |s-1|^M, \tag{3.16}$$

(with a constant $b < \frac{1}{b_0}$), on the compact disk $|s-1| \le b_0$. Thus

$$\begin{aligned} I^*_{k,\kappa}(w^N) = \sum_{j=1}^{M-1} \gamma_j \left(\frac{1}{2\pi i} \int_{\mathcal{C}_0} (s-1)^j \,(\log(s-1))^k \, w^s \, \mathrm{d}s \right) + \\ + \frac{1}{2\pi i} \int_{\mathcal{C}_0} r_M(s) \,(\log(s-1))^k \, w^s \, \mathrm{d}s \,. \end{aligned} \tag{3.17}$$

In the main term, we substitute $s \to s+1$ and then replace $\mathcal{C}_0 - 1$ by the classical "Hankel loop" $\mathcal{H}$ which consists of the lower edge of the real line from $-\infty$ to $-b_0$ (say), of the circle $\mathcal{C}_0 - 1$, and of the upper edge of the reals from $-b_0$ to $-\infty$. Using the well-known formula

$$\frac{1}{2\pi i} \int_{\mathcal{H}} e^s \, s^a (\log s)^k \, \mathrm{d}s = \frac{\mathrm{d}^k}{\mathrm{d}a^k} \left(\frac{1}{\Gamma(-a)} \right) \qquad (a \in \mathbb{C}, \ k \in \mathbb{N}_0),$$

we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathcal{H}} s^j (\log s)^k w^{s+1} \, \mathrm{d}s = \\ &= w(\log w)^{-j-1} \left(\frac{1}{2\pi i} \int_{\mathcal{H}} e^s \, s^j (\log s - \log\log w)^k \, \mathrm{d}s \right) = \\ &= w(\log w)^{-j-1} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} (\log\log w)^{k-r} \, \frac{\mathrm{d}^r}{\mathrm{d}a^r} \left(\frac{1}{\Gamma(-a)} \right) \Big|_{a=j} = \\ &= w(\log w)^{-j-1} \mathcal{P}^*_k(\log\log w), \end{aligned} \tag{3.18}$$

where $\mathcal{P}^*_k(\cdot)$ is a polynomial of degree $k-1$. Estimating the error we thereby commit, we get

$$\begin{aligned} &\gamma_j w \int_{b_0}^{\infty} u^j (1 + |\log u|^k) w^{1-u} \, \mathrm{d}u \ll \\ &\ll \gamma_j w (\log w)^{-j-1} (\log\log w)^k \int_{b_0 \log w}^{\infty} e^{-u} u^j \left(1 + |\log u|^k \right) \, \mathrm{d}u \ll \\ &\ll \gamma_j b_0^j w^{1-b_0} (\log w)^{-1} (\log\log w)^{2k}, \end{aligned}$$

uniformly in $1 \le j \le M(w^N) - 1$. Here we have used Lemma 5. Note that (2.10) is certainly satisfied for w sufficiently large in view of (1.2). Summing over $j = 1, \dots, M(w^N) - 1$ and appealing to the first clause of (3.16), we see that the total contribution of these error terms is $O(w^{1-b_0})$, hence "harmless".

We finally estimate the remainder integral in (3.17): Again we substitute $s \to s+1$ and then replace the path of integration $\mathcal{C}_0 - 1$ by a new contour consisting of a circle $\mathcal{C}(\varepsilon)$ around the origin with radius

$$\varepsilon = \varepsilon(w) = b_5 (\log w)^{-2/5} (\log\log w)^{-6/5} \tag{3.19}$$

(starting and ending at the point $-\varepsilon$), and of two straight line segments joining the points $-b_0$ and $-\varepsilon$, on the lower and upper edge of the real line, respectively.

The contribution of these line segments to the remainder integral in (3.17) is, by (3.16),

$$\begin{aligned} &\ll b^M\, w \int_\varepsilon^{b_0} w^{-u} u^M |\log u|^k \,\mathrm{d}u \ll \\ &\ll b^M\; w (\log w)^{-M-1} \int_{\varepsilon \log w}^{\infty} e^{-u}\, u^M\; \left(|\log u|^k + (\log\log w)^k \right) \mathrm{d}u \,. \end{aligned}$$

To this last integral we again apply Lemma 5. In view of (1.2) and (3.19), condition (2.10) is satisfied (for appropriate $b_5 > 0$), and we see that the above expression is

$$\ll (b\varepsilon)^M\, w^{1-\varepsilon} (\log w)^{-1} (\log\log w)^k \,. \tag{3.20}$$

Similarly, again using (3.16), we get

$$\begin{aligned} &\int_{\mathcal{C}(\varepsilon)} r_M(1+s) (\log s)^k w^{1+s} \,\mathrm{d}s \ll \\ &\ll w^{1+\varepsilon}\, |\log \varepsilon|^k b^M \varepsilon^{M+1} \ll w^{1+\varepsilon}\, (\log\log w)^k b^M \varepsilon^{M+1} \,. \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we thus obtain

$$\int_{\mathcal{C}_0} r_M(s)\, (\log(s-1))^k\, w^s \,\mathrm{d}s \ll w^{1+\varepsilon} (b\varepsilon)^M \,. \tag{3.22}$$

Appealing to (1.2) and (3.19) one more time, we see that

$$\begin{aligned} \log(w^\varepsilon (b\varepsilon)^M) &\le b_5 (\log w)^{3/5} (\log\log w)^{-6/5} + \\ &+ [c^* (\log(w^N))^{3/5} (\log\log(w^N))^{-6/5}] \times \\ &\times \left(\log b + \log b_5 - \frac{2}{5} \log\log w - \frac{6}{5} \log\log\log w \right) \le \\ &\le -\frac{c^*}{5} (\log w)^{3/5} (\log\log w)^{-1/5} \end{aligned}$$

for w sufficiently large. Consequently, the right-hand side of (3.22) is $O(w\,\delta_{11}(w))$. Replacing finally w by $x^{1/N}$, we complete the proof of Lemma 6.

Summation over all decompositions $\mathbf{I}$ of y gives the assertion of the Theorem — apart from the bound $O((b_* l)^l)$ for the coefficients of $\mathcal{P}_l$. To establish the latter, a close look at the proof of Lemma 6 (in particular at (3.16) and (3.18)) shows that the only nontrivial step is to estimate

$$\frac{\mathrm{d}^r}{\mathrm{d}a^r}\left(\frac{1}{\Gamma(-a)}\right)\bigg|_{a=j}$$

(where $j = l-1,\ r \le d_*$). By the functional equation and (a crude version of) Stirling's formula, this is

$$= \frac{\mathrm{d}^r}{\mathrm{d}a^r}\left(-\frac{1}{\pi}\sin(\pi a)\Gamma(1+a)\right)\bigg|_{a=j} \ll \sum_{i=0}^{r} |\Gamma^{(i)}(1+j)| \ll (b_{**}l)^l\,.$$

□

Remark. It follows from this bound for the coefficients of the $\mathcal{P}_l$ that, for $l \le M(x)$, and x sufficiently large,

$$\begin{aligned}\frac{\mathcal{P}_l(\log\log x)}{(\log x)^l} &\ll (b_* l)^l\,\frac{(\log\log x)^{d_*-1}}{(\log x)^l} \le \\ &\le \exp\left(l(\log b_* + \log M) + (d_*-1)\log\log\log x - l\log\log x\right) \le \\ &\le \exp\left(-\tfrac{l}{5}\log\log x\right) = (\log x)^{-l/5}\,.\end{aligned}$$

Consequently, for any fixed $M^* \in \mathbb{N}$,

$$\begin{aligned}&\sum_{l=M^*+1}^{M}\frac{\mathcal{P}_l(\log\log x)}{(\log x)^l} \ll_{M^*} \\ &\ll_{M^*} (\log\log x)^{d_*-1}\sum_{l=M^*+1}^{5(M^*+1)}(\log x)^{-l} + \sum_{5M^*+6}^{M}(\log x)^{-l/5} \ll \\ &\ll \frac{(\log\log x)^{d_*-1}}{(\log x)^{M^*+1}},\end{aligned}$$

hence our Theorem implies that

$$A_y(x) = C_y x + x^{1/N}\sum_{l=2}^{M^*}\frac{\mathcal{P}_l(\log\log x)}{(\log x)^l} + O\Big(x^{1/N}\frac{(\log\log x)^{d_*-1}}{(\log x)^{M^*+1}}\Big), \tag{3.23}$$

for any fixed $M^* \in \mathbb{N}$, where the O-constant depends on M^*.

4. Applications: Counting functions of certain algebraic structures

As we pointed out in the introduction, the function $a(n)$ which counts the number of non-isomorphic Abelian groups of order n is the classic example for the class of arithmetic functions under consideration. In this important case,

$$(\lambda_n)_{n\in\mathbb{N}_0} = (P(n))_{n\in\mathbb{N}_0} = (1,1,2,3,5,7,11,15,\dots) \tag{3.24}$$

($P(\cdot)$ the partition function), thus $N = 2$, $\mathcal{Y} \subseteq \mathbb{N}$, and d_* is the maximal integer such that $2^{d_*}|y$. Our Theorem contains the sharpest result known todate — the order term seems to be "definitive" on the basis of our present knowledge about zero-free regions of the zeta-function. In particular, it improves upon the work of Krätzel and Wolke [7] who established the counterpart of our formula (3.23) and indicated that their method actually yields an error term $O(x^{1/2}\exp(-c'(\log x)^{1/3}(\log\log x)^{-1/3}))$. In addition, they *conjectured* that an improvement up to $O(x^{1/2}\exp(-c''(\log x)^{3/5}(\log\log x)^{-1/5}))$ should be in reach of present methods.

A related arithmetic function is $S(n)$ which counts the number of (isomorphism classes of) semisimple rings with n elements. Its generating Dirichlet series is

$$\sum_{n=1}^{\infty}\frac{S(n)}{n^s} = \prod_{m\in\mathbb{N}}\prod_{k\in\mathbb{N}}\zeta(m^2k\,s) \qquad (\operatorname{Re}s>1)\,. \tag{4.1}$$

(See, e.g., Ivić [3, p. 38], or Kühleitner [8] where the algebraic background is sketched.) It is clear by (4.1) that $S(n) = a(n)$ for all $n \in \mathbb{N}$ which are free of fourth powers. (These are about 92,39% of all positive integers.) Actually, the function $S(n)$ was not studied a lot in the literature because its generating function shares practically all useful analytic properties with

$$\prod_{k\in\mathbb{N}}\zeta(k\,s),$$

the generating function of $a(n)$. (In particular, our Theorem immediately applies to $S(n)$ and yields an asymptotic formula which is identical with that for $a(n)$, apart from the value of C_y and the coefficients of the polynomials $\mathcal{P}_l$.)

Quite recently, Kühleitner [8] had the idea to compare $S(n)$ with $a(n)$ by studying power moments of the ratio $\varrho(n) = \frac{S(n)}{a(n)}$. For any $r > 0$, he proved that

$$\sum_{n\le x}(\varrho(n))^{-r} = A^{(r)}\,x + x^{1/4}\sum_{k=0}^{M(x)}A_k^{(r)}(\log x)^{-\tau-1-k} + O(x^{1/4}\delta(x))$$

where $M(x)$ and $\delta(x)$ are as in our Theorem, and $\tau = 1-(\frac{5}{6})^r$.

We turn our attention to the value distribution of this function $\varrho(n)$. Obviously it is multiplicative, prime-independent and ≥ 1 throughout. We note that, for an arbitrary prime p,

$$(S(p^n))_{n\in\mathbb{N}_0} = (P^*(n))_{n\in\mathbb{N}_0} = (1,1,2,3,6,8,13,18,\dots),$$

where $P^*(n)$ is generated by

$$\begin{aligned}\sum_{n=0}^{\infty} P^*(n) z^n &= \prod_{m\in\mathbb{N}} \prod_{k\in\mathbb{N}} \left(1 - z^{m^2 k}\right)^{-1} = \\ &= \prod_{m\in\mathbb{N}} \left(\sum_{n=0}^{\infty} P(n) z^{m^2 n}\right) \qquad (|z|<1)\,.\end{aligned} \tag{4.2}$$

Therefore, the sequence of λ's corresponding to $(\varrho(n))_{n\in\mathbb{N}}$ is

$$(\lambda_n)_{n\in\mathbb{N}_0} = \left(\frac{P^*(n)}{P(n)}\right)_{n\in\mathbb{N}_0} = (1,1,1,1,\tfrac{6}{5},\tfrac{8}{7},\tfrac{13}{11},\tfrac{6}{5},\dots)\,.$$

It is clear that $N=4$, since, by (4.2), $P^*(n) > P(n)$ for $n \geq 4$. $\mathcal{Y}$ is a discrete semigroup of rationals ≥ 1. To verify our conditions (1.1), it remains to show that $\lambda_n \to \infty$. Let $K_0 \in \mathbb{N}$ be fixed, then it follows from (4.2) that, for n sufficiently large,

$$P^*(n) \geq \sum_{\substack{(m,l)\in\mathbb{N}_0^2:\\ m+4l=n}} P(m)\,P(l) \geq \sum_{l=0}^{K_0} P(n-4l)\,.$$

In view of the classic asymptotic

$$P(n) \sim \frac{4\sqrt{3}}{n} \exp\left(\pi\sqrt{\frac{2}{3}\,n}\right),$$

this implies that

$$\liminf_{n\to\infty} \lambda_n = \liminf_{n\to\infty} \left(\frac{P^*(n)}{P(n)}\right) \geq K_0$$

which can be chosen arbitrarily large. Therefore, our Theorem applies and it gives, for each $y \in \mathcal{Y}$,

$$A_y(x) = C_y x + x^{1/4} \sum_{l=2}^{M(x)} \frac{\mathcal{P}_l(\log\log x)}{(\log x)^l} + O(x^{1/4}\delta_0(x)),$$

where $\mathcal{P}_l$ are polynomials of degree $\leq d_* - 1$, and d_* is the maximal integer such that $(\frac{6}{5})^{-d_*}\, y \in \mathcal{Y}$. (Again, the whole sum over l vanishes if $(\frac{6}{5})^{-1}\, y \notin \mathcal{Y}$.)

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