

A note on regularly asymptotic points

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Abstract. A condition of Schmets and Valdivia for a boundary point of a domain in the complex plane to be regularly asymptotic is ameliorated.

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Introduction

Using the notation by Schmets and Valdivia [2], we denote by Ω a non-void domain contained in the complex plane \mathbb{C} , by D a non-void subset of its boundary $\partial\Omega$. Throughout this paper we suppose that D is finite.

Definition. We say that a holomorphic function f on Ω has an asymptotic expansion at a boundary point $u \in \partial\Omega$ if for every $n = 0, 1, 2, \dots$ the limit

$$(1) \quad \lim_{\substack{z \in \Omega \\ z \rightarrow u}} f^{[n]}(z, u) = a_n \in \mathbb{C}$$

exists, where the functions $f^{[n]}$ are defined by induction

$$(2) \quad \begin{aligned} f^{[0]}(z, u) &= f(z), \\ f^{[n+1]}(z, u) &= \frac{f^{[n]}(z, u) - a_n}{z - u}. \end{aligned}$$

So, in fact, we have

$$\lim_{\substack{z \in \Omega \\ z \rightarrow u}} \frac{f(z) - \sum_{j=0}^n a_j (z - u)^j}{(z - u)^{n+1}} = a_{n+1} \quad (\forall n = 0, 1, 2, \dots).$$

We put $f^{[n]}(u) = a_n$. We say that the series $\sum_{n=0}^{\infty} a_n (z - u)^n$ is the asymptotic expansion of f at u and write

$$f(z) \approx \sum_{n=0}^{\infty} a_n (z - u)^n \text{ at } u.$$

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The set of all holomorphic functions on Ω having an asymptotic expansion at every point $u \in D$ is denoted by $\mathcal{A}(\Omega; D)$.

We say that D is regularly asymptotic for Ω if, for every family of complex numbers $\{a_{u,n}; u \in D, n = 0, 1, 2, \dots\}$, there is a function $f \in \mathcal{A}(\Omega; D)$ such that

$$f(z) \approx \sum_{n=0}^{\infty} a_{u,n}(z-u)^n \quad \text{at } u$$

for every $u \in D$.

The aim of this paper is to generalize the following sufficient condition for D to be regularly asymptotic for Ω (Theorem 1). We give also a condition implying that a boundary point is not regularly asymptotic (Theorem 2).

Theorem ([2, Theorem 3.7]). *A finite set $D \subset \partial\Omega$ is regularly asymptotic for Ω if every point $u \in D$ has the following property: there are connected subsets $A_k \subset \mathbb{C} \setminus \Omega$ ($k = 1, 2, \dots$) and $u \neq v_k \in A_k$ such that*

$$\lim_{k \rightarrow \infty} v_k = u, \quad \lim_{k \rightarrow \infty} \frac{\text{diam } A_k}{|v_k - u|} = \infty.$$

As a consequence, a point $u \in \partial\Omega$ is regularly asymptotic for Ω if it belongs to a component of $\mathbb{C} \setminus \Omega$ containing more than one point.

Schmets and Valdivia [2] proved this theorem using the following

Proposition ([2, Proposition 3.6]). *A finite subset D of Ω is regularly asymptotic for Ω iff the following condition is satisfied: there is $r > 0$ such that for every compact subset $K \subset \Omega$ and $u \in D$, there is an integer $p \in \mathbb{N}$ such that, for every $h > 0$, there is a function $f \in \mathcal{A}(\Omega; D)$ verifying*

$$|f(z)| \leq 1 \quad \text{for all } z \in K \cup \left(\bigcup_{u' \in D} \{z' \in \Omega; |z' - u'| \leq r\} \right)$$

and

$$|f^{[p]}(u)| > h.$$

For proving the theorem, the authors applied the proposition with $p = 1$ and $f(z)$ equal to a multiple of a determination of $\sqrt{(z - v_k)(z - w_k)}$, $v_k, w_k \in A_k$. Using a higher p , we can generalize the cited result.

Generalization

Theorem 1. *A finite set $D \subset \partial\Omega$ is regularly asymptotic for Ω if every point $u \in D$ has the following property:*

there are connected subsets A_k of $\mathbb{C} \setminus \Omega$ ($k = 1, 2, \dots$), $u \neq v_k \in A_k$ and $q > 0$ such that

$$(3) \quad \lim_{k \rightarrow \infty} v_k = u,$$

and

$$(4) \quad \text{diam } A_k > |v_k - u|^q.$$

PROOF: Without loss of generality we can suppose that

$$(5) \quad |v_k - u| < \frac{1}{2}$$

and

$$(6) \quad q \geq 2.$$

If we replace A_k with a convenient connected closed subset of A_k , we can have, besides (4) and other hypotheses, in addition

$$(7) \quad \text{diam } A_k < 2|v_k - u|^q.$$

This implies that $\text{diam } A_k < |v_k - u|$, hence A_k does not contain the point u . As D is finite and $\lim \text{diam } A_k = 0$, we have $D \cap A_k = \emptyset$ for k large enough. If we choose an integer

$$(8) \quad p \geq q + 1 \geq 3,$$

we have by (4), (5) and (8)

$$\text{diam } A_k > |v_k - u|^{q-p-\frac{1}{4}} \cdot |v_k - u|^{p+\frac{1}{4}} > 2|v_k - u|^{p+\frac{1}{4}}.$$

As A_k is connected, it follows that we can choose a point $w_k \in A_k$ satisfying

$$(9) \quad |w_k - v_k| = |v_k - u|^{p+\frac{1}{4}}.$$

Thus, by (3) and (8) we have $\lim_{k \rightarrow \infty} w_k = u$, moreover

$$(10) \quad \lim_{k \rightarrow \infty} \frac{w_k - u}{v_k - u} = \lim_{k \rightarrow \infty} \frac{(v_k - u) + (w_k - v_k)}{v_k - u} = 1.$$

Denote by g_k a determination of the analytic function $\sqrt{(\bullet - v_k)(\bullet - w_k)}$ defined on $\mathbb{C} \setminus A_k$. Consequently, g_k is defined on Ω and belongs to $\mathcal{A}(\Omega; D)$ for k large enough. Evidently, for $k = 1, 2, \dots$, the functions $|g_k|$ are bounded on the bounded set

$$K \cup \left(\bigcup_{u' \in D} \{z' \in \Omega; |z' - u'| \leq r\} \right)$$

by a constant C independent on k . We will apply the cited proposition with the functions $f_k := \frac{g_k}{C}$ and with $2p$ instead of p . The function g_k , being holomorphic at the point u , has its asymptotic expansion equal to the Taylor expansion at u ; so $f_k^{[2p]}(u) = \frac{1}{(2p)!} f_k^{(2p)}(u)$ and the result will follow from the Proposition if we prove

$$(11) \quad \lim_{k \rightarrow \infty} |g_k^{(2p)}(u)| = \infty.$$

To this end, fix an index k and denote

$$(12) \quad f_\alpha(z) := (z - v_k)^\alpha (z - w_k)^\alpha.$$

It can be verified by a direct calculation that

$$(13) \quad f_\alpha''(z) = \alpha(\alpha - 1)f_{\alpha-2}(z)(v_k - w_k)^2 + 2\alpha(2\alpha - 1)f_{\alpha-1}(z).$$

The meaning of this equality between multi-valued functions is as follows: if f_α in the formula (13) signifies a determination of (12), then (13) holds for

$$f_{\alpha-1}(z) = \frac{f_\alpha(z)}{(z - v_k)(z - w_k)}, \quad \text{and} \quad f_{\alpha-2}(z) = \frac{f_\alpha(z)}{(z - v_k)^2(z - w_k)^2}.$$

For $\alpha = \frac{1}{2}$, the coefficient $2\alpha(2\alpha - 1)$ equals zero, but if we calculate higher derivatives of even order of the function $f_{\frac{1}{2}}$ using recurrence relation (13), we do not meet in (13) other zero coefficients. Thus

$$(14) \quad f_{\frac{1}{2}}''(z) = -\frac{1}{4}f_{-\frac{3}{2}}(z)(v_k - w_k)^2$$

and from (13) follows by induction

$$(15) \quad f_{\frac{1}{2}}^{(2p)}(z) = \sum_{j=1}^p \alpha_j f_{\frac{1}{2}-p-j}(z)(v_k - w_k)^{2j}$$

with $\alpha_j \in \mathbb{R}$ depending only on j and p , $\alpha_1 \neq 0$. By (12) it follows

$$f_{\frac{1}{2}}^{(2p)}(u) = \sum_{j=1}^p \alpha_j (u - v_k)^{\frac{1}{2}-p-j} (u - w_k)^{\frac{1}{2}-p-j} (v_k - w_k)^{2j} = C_k \sum_{j=1}^p B_{k,j},$$

where

$$C_k = \alpha_1(u - v_k)^{-1-2p} \cdot (v_k - w_k)^2$$

and

$$B_{k,j} = \frac{\alpha_j}{\alpha_1} \cdot \frac{(u - w_k)^{\frac{1}{2}-p-j}}{(u - v_k)^{\frac{1}{2}-p-j}} \cdot \frac{(v_k - w_k)^{2j-2}}{(u - v_k)^{2j-2}}.$$

Now we pass to the limit. By (9) and (3) we have

$$\lim_{k \rightarrow \infty} |C_k| = \lim_{k \rightarrow \infty} \alpha_1 |v_k - u|^{-1-2p+2p+\frac{1}{2}} = \infty$$

and by (10), (9), (3) and (8), we have

$$\lim_{k \rightarrow \infty} B_{k,1} = 1, \quad \lim_{k \rightarrow \infty} B_{k,j} = 0 \quad \text{for } j \geq 2.$$

This proves the relation (11) and consequently the theorem. □

Now we will consider a domain Ω of the form

$$(16) \quad \Omega = \tilde{\Omega} \setminus \left(\{u\} \cup \bigcup_{k=1}^{\infty} A_k \right)$$

where $\tilde{\Omega}$ is a domain including the point u and A_k are disjoint closed subsets of $\tilde{\Omega} \setminus \{u\}$ with $\lim_{k \rightarrow \infty} \text{dist}(A_k, u) = 0$.

Theorem 2. *Suppose that there are points $v_k \in A_k$ with $\lim v_k = u$ and numbers $R_k > \text{diam } A_k$ for which the set*

$$G = \bigcup_{k=1}^{\infty} \{z; |z - v_k| < R_k\} \cup \{u\}$$

is not neighbourhood of the point u and

$$(17) \quad \sum_{k=1}^{\infty} \frac{\text{diam } A_k}{R_k^q} < \infty \quad \text{for every } q \geq 0.$$

Then the point u is not regularly asymptotic for the domain Ω .

PROOF: At first, we need some preparation and auxiliary claims. As the set G is not neighbourhood of zero, there are points $z_m \in \Omega$ ($m \in \mathbb{N}$) with

$$(18) \quad z_m \neq u, \quad \lim z_m = u \quad \text{and} \quad |v_k - z_m| \geq R_k$$

for all $m, k \in \mathbb{N}$. Consequently,

$$(19) \quad |v_k - u| \geq R_k$$

and thanks to $\lim v_k = u$ we obtain by reindexation

$$(20) \quad R_k \searrow 0.$$

Let us put $d_k = \text{diam } A_k + e^{-\frac{k}{R_k}}$, denote by D_k the disk $\{z; |z - v_k| \leq d_k\}$ and by ∂D_k its boundary circle $\{z; |z - v_k| = d_k\}$ counter-clockwise oriented. Then

$$A_k \subset \text{int } D_k$$

and by (17)

$$(21) \quad \sum_{k=1}^{\infty} \frac{d_k}{R_k^q} < \infty$$

for all $q \geq 0$. We can suppose

$$(22) \quad \sum_{k=1}^{\infty} \frac{d_k}{R_k} < \frac{1}{4};$$

otherwise we replace $\tilde{\Omega}$ with $\tilde{\Omega} \setminus \bigcup_{k=1}^l A_k$ for a convenient l . Then by (19) and (22) the distance of D_k from the point u is

$$(23) \quad |v_k - u| - d_k \geq R_k - d_k \geq R_k - \frac{1}{4}R_k = \frac{3}{4}R_k.$$

Claim 1. For any $R > 0$ there is a circle

$$\kappa_\varrho := \{z; |z - u| = \varrho\} \subset \tilde{\Omega} \setminus \bigcup_{k=1}^{\infty} D_k$$

with $0 < \varrho < R$.

Let us observe that only relations (19), (20), (22) are needed for the proof of this claim.

PROOF: Choose a k' for which

$$(24) \quad R_{k'} < R.$$

By (23) (deduced from (19) and (22)) and (20), for $k \leq k'$, we have

$$|v_k - u| - d_k \geq \frac{3}{4}R_{k'}.$$

Consequently, the disks D_k ($k = 1, 2, \dots, k'$) do not meet the disk $\{z; |z - u| \leq \frac{1}{2}R_{k'}\}$. On the other hand, for $k > k'$ the disk D_k is contained in the annulus

$$(25) \quad \{z; |v_k - u| - d_k \leq |z - u| \leq |v_k - u| + d_k\}$$

of the width $2d_k$. By (20) and (22), the sum of the widths is

$$\sum_{k=k'+1}^{\infty} 2d_k \leq R_{k'} \sum \frac{2d_k}{R_k} < \frac{1}{2}R_{k'},$$

hence the sets (25) cannot cover the set $\{z; 0 < |z - u| \leq \frac{1}{2}R_{k'}\}$ and the claim is proved. \square

Let f be a holomorphic function on Ω having an asymptotic expansion at the point u with coefficients a_n ($n = 0, 1, \dots$). We will prove that u is not regularly asymptotic showing that the coefficients cannot be (cf. (21))

$$(26) \quad a_n = n^n + 4^{n+1} \cdot \sum_{k=1}^{\infty} \frac{d_k}{R_k^{n+1}}.$$

Due to Claim 1, choose circles κ_{ϱ_j} ($j = 1, 2, \dots$) contained in Ω and disjoint with disks D_k (for each $k, j \in \mathbb{N}$),

$$(27) \quad \varrho_j \searrow 0, \varrho_j > \varrho_{j+1}.$$

As the limit $\lim_{z \rightarrow u, z \in \Omega} f(z) = a_0$ exists, we can suppose that ϱ_1 is so small that for some b we have

$$(28) \quad |f(z)| \leq b \text{ whenever } z \in \Omega, |z - u| \leq \varrho_1$$

and that

$$\{z; |z - u| \leq \varrho_1\} \subset \tilde{\Omega}.$$

Let N_j be the set of the indexes $k \in \mathbb{N}$ for which

$$D_k \subset \{z; \varrho_{j+1} < |z - u| < \varrho_j\}.$$

Then N_j is finite; denote by γ_j the boundary cycle of the set $\bigcup_{k \in N_j} D_k$ directed so that the interior of $\bigcup_{k \in N_j} D_k$ lies to the left of γ_j . γ_j is the sum of arcs of the circles ∂D_k , is situated in Ω and satisfies

$$\{z; \text{ind}_{\gamma_j} z = 1\} = \text{int} \bigcup_{k \in N_j} D_k.$$

Hence the cycle $\kappa_1 - \gamma_1 - \dots - \gamma_J - \kappa_{J+1}$ ($J \in \mathbb{N}$) is homologous with zero in Ω , so we can use the Cauchy formula below. Namely, by (18) and (22) the point z_m does not belong to any disk D_k . For m large enough we have $|z_m - u| < \varrho_1$, then for J large enough we have $\varrho_{J+1} < |z_m - u|$ and thus

$$f(z_m) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)d\zeta}{\zeta - z_m} - \sum_{j=1}^J \int_{\gamma_j} \frac{f(\zeta)d\zeta}{\zeta - z_m} - \int_{\kappa_{J+1}} \frac{f(\zeta)d\zeta}{\zeta - z_m} \right].$$

Thanks to (27) and (28), we have $\lim_{J \rightarrow \infty} \int_{\kappa_{J+1}} \frac{f(\zeta)d\zeta}{\zeta - z_m} = 0$, so

$$(29) \quad f(z_m) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)d\zeta}{\zeta - z_m} - \sum_{j=1}^{\infty} \int_{\gamma_j} \frac{f(\zeta)d\zeta}{\zeta - z_m} \right].$$

Claim 2. *If m is as large as $|z_m - u| < \varrho_1$, then for $n = 0, 1, 2, \dots$, we have*

$$(30) \quad f^{[n]}(z_m, u) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)d\zeta}{(\zeta - z_m)(\zeta - u)^n} - \sum_{j=1}^{\infty} \int_{\gamma_j} \frac{f(\zeta)d\zeta}{(\zeta - z_m)(\zeta - u)^n} \right]$$

and

$$(31) \quad a_n = \lim_{m \rightarrow \infty} f^{[n]}(z_m, u) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)d\zeta}{(\zeta - u)^{n+1}} - \sum_{j=1}^{\infty} \int_{\gamma_j} \frac{f(\zeta)d\zeta}{(\zeta - u)^{n+1}} \right].$$

PROOF: We shall proceed by induction. First we deduce the formula (31) from (30) using Lebesgue majorization theorem. As any point ζ of a cycle γ_j belongs to ∂D_k for some k , we have by (28), (18), definition of ∂D_k , (19) and (23)

$$\begin{aligned} & \left| \frac{f(\zeta)}{(\zeta - z_m)(\zeta - u)^n} \right| = \left| \frac{f(\zeta)}{(\zeta - v_k - (z_m - v_k))(\zeta - u)^n} \right| \\ & \leq \frac{b}{(R_k - d_k)(|v_k - u| - d_k)^n} \leq \frac{b}{(R_k - d_k)^{n+1}} \leq \left(\frac{4}{3}\right)^{n+1} \frac{b}{R_k^{n+1}}. \end{aligned}$$

Hence the function g defined by $g(\zeta) = \left(\frac{4}{3}\right)^{n+1} \frac{b}{R_k^{n+1}}$ for $\zeta \in \partial D_k \setminus \bigcup_{k'=1}^{k-1} \partial D_{k'}$ is a majorant. Thanks to (21), it is integrable even on the set

$$(32) \quad \bigcup_{k=1}^{\infty} \partial D_k \supset \bigcup_{j=1}^{\infty} \gamma_j$$

with respect to the length measure. Hence the implication (30) \Rightarrow (31) is proved.

Induction: If we put $n = 0$, the formula (30) turns into the Cauchy formula (29). Using the recurrent definition (cf. (2))

$$f^{[n+1]}(z_m, u) = \frac{f^{[n]}(z_m, u) - a_n}{z_m - u},$$

we deduce easily the formula (30) for $n + 1$ from (30) and (31) and the claim is proved. \square

Now we complete the proof of the theorem. Integrating in (31) along $\bigcup_{k=1}^{\infty} \partial D_k$ instead of $\bigcup_{j=1}^{\infty} \gamma_j$, we obtain by (32), (28) and (23)

$$\begin{aligned} |a_n| &\leq \frac{1}{2\pi} \cdot \left[2\pi \varrho_1 \frac{b}{\varrho_1^{n+1}} + \sum_{k=1}^{\infty} 2\pi d_k \frac{b}{(|v_k - u| - d_k)^{n+1}} \right] \\ &\leq \frac{b}{\varrho_1^n} + \left(\frac{4}{3}\right)^{n+1} b \cdot \sum_{k=1}^{\infty} \frac{d_k}{R_k^{n+1}}, \end{aligned}$$

which cannot be true for all n together with (26).

Corollary. *Suppose the domain Ω to be of the form (16) with*

$$(33) \quad \sum (\text{dist}(A_k, u))^p < \infty$$

for some $p > 0$. If, for every $q \geq 0$,

$$(34) \quad \text{diam } A_k \leq (\text{dist}(A_k, u))^q$$

except a finite number (depending on q) of indexes k , then the point u is not regularly asymptotic.

PROOF: Choose points $v_k \in A_k$ so that $\text{dist}(A_k, u) = |v_k - u|$. Hence, except a finite number of indexes k ,

$$(35) \quad \text{diam } A_k \leq |v_k - u|^q.$$

Thanks to (33), we can suppose without loss of generality that

$$(36) \quad \sum_{k=1}^{\infty} |v_k - u|^p < \frac{1}{4}.$$

So, putting for a moment $d_k = |v_k - u|^{p+1}$ and $R_k = |v_k - u|$, we have

$$\sum_{k=1}^{\infty} \frac{d_k}{R_k} < \frac{1}{4},$$

which is the relation (22). Also the relation (20) can be satisfied by reindexation and we can apply Claim 1 affirming that there are circles κ_ϱ with arbitrarily small ϱ , disjoint with disks $\{z; |z - v_k| \leq |v_k - u|^{p+1}\}$. Now we change the notation putting $R_k = |v_k - u|^{p+1}$. By this way we see that, for any $R > 0$, there is a circle κ_ϱ , $0 < \varrho < R$ disjoint with $\{z; |z - v_k| \leq R_k\}$. It verifies the hypothesis of Theorem 2 that G is not neighbourhood of the point u . Now, choose a $q \geq 0$. By (35) we have

$$\text{diam } A_k \leq |v_k - u|^{q(p+1)+p}$$

except a finite number of indexes k . It follows by the last definition of R_k and by (36) that

$$\sum_{k=1}^{\infty} \frac{\text{diam } A_k}{R_k^q} < \infty$$

and Theorem 2 gives the result. \square

Remark. Suppose that for the domain Ω of the form (16) the hypothesis (33) of the preceding corollary is satisfied. If in addition the sets A_k are connected, the preceding corollary with Theorem 1 show that the relation (34) characterizes that the point u is not regularly asymptotic. Indeed, if for some q the relation (34) is not satisfied for an infinite number of indexes k , we obtain the hypothesis (4) of Theorem 1 for a suitable subsequence of $\{A_k\}$.

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