Nowhere dense subsets and Booth's Lemma

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Abstract. The following statement is proved to be independent from [LB $+\neg$ CH]: $(*)$ Let X be a Tychonoff space with $c(X) \leq \aleph_0$ and $\pi w(X) < \mathfrak{C}$. Then a union of less than $\mathfrak{C}$ of nowhere dense subsets of X is a union of not greater than $\pi w(X)$ of nowhere dense subsets.

Keywords: nowhere dense subset, Booth's Lemma, π-weight

Classification: Primary 54A35, 03E35; Secondary 54A25

1. Introduction

It is known that under Booth's Lemma LB in a space with a countable π-base a union less than $\mathfrak{C}$ of nowhere dense subsets is a meager set (see, for instance, [1]), in another words, the ideal of meager sets is $(< \mathfrak{C})$-additive.

Let us remind that Booth's Lemma LB (called also the combinatorial Principle $P(\mathfrak{C})$) means:

LB (or $P(\mathfrak{C})$) *Let $\mathcal{A}$ be a family of infinite subsets of a countable set and $|\mathcal{A}| < \mathfrak{C}$. If the intersection of each infinite subfamily of $\mathcal{A}$ is infinite, then there exists an infinite subset B, such that $|B \setminus A| < \aleph_0$ for every $A \in \mathcal{A}$.*

L.B. Shapiro suggested that π-weight of a space plays the important role here, and raised the following question:

Is the following assertion $(*)$ true under LB:

$(*)$ *Let X be a Tychonoff space with $c(X) \leq \aleph_0$ and $\pi w(X) = \tau < \mathfrak{C}$. Then a union of less than $\mathfrak{C}$ of nowhere dense subsets of X is a union of not greater than τ of nowhere dense subsets.*

The answer to Shapiro's question turned out to be interesting. In this answer the following assertion occurs:

$(**)$ *Let X be a Tychonoff space with $c(X) \leq \aleph_0$ and $\pi w(X) = \tau < \mathfrak{C}$. Then a union of less than $\mathfrak{C}$ of nowhere dense subsets of X is a union of not more than τ^+ of nowhere dense subsets.*

2. Cardinal function $Cov_{\aleph_0}$

We are introducing the following cardinal function $Cov_{\aleph_0}$.

Let $[m]^{\leq\omega}$ be the family of all countable subsets of a cardinal m. A family $\mathcal{B} \subseteq [m]^{\leq\omega}$ is said to be a covering base of the ideal $[m]^{\leq\omega}$ if for every $A \in [m]^{\leq\omega}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

2.1 Definition. $Cov_{\aleph_0}(m)$ is the minimum of powers of covering bases of the ideal $[m]^{\leq\omega}$.

2.2 Theorem. *Under LB the following assertion is true:*
Let X be a Tychonoff space with $c(X) \leq \aleph_0$. Then a union of less than $\mathfrak{C}$ of nowhere dense subsets of X is a union of not greater than $Cov_{\aleph_0}(\pi w(X))$ of nowhere dense subsets.

Proof: As X is a Tychonoff space, so $X \subseteq I^\nu$ for some cardinal ν. Let $\pi w(X) = \tau$. Without loss of generality it can be supposed that in X there exists a π-base $\mathcal{V}$ of the power τ and such that $\pi_\tau^{-1}\pi_\tau V = V$ for every $V \in \mathcal{V}$ (π_τ is the projection of X into I^τ).

As $c(X) \leq \aleph_0$, for every nowhere dense subset $N \subset X$ there exists a countable subset $A \subseteq \tau$ such that $N \subseteq \pi_A^{-1}\pi_A N$. We will denote such a subset A by A_N.

So, let $\mathcal{N}$ be a family of power less than $\mathfrak{C}$ of nowhere dense subsets of X. For the family $\mathcal{A} = \{A_N : N \in \mathcal{N}\}$ let us find some covering family $\mathcal{B}$ of power not greater than $Cov_{\aleph_0}(\tau)$. For every $B \in \mathcal{B}$ let $\mathcal{N}_B = \{N \in \mathcal{N} : A_N \subseteq B\}$. As B is countable, so $\pi_B(\cup\mathcal{N}_B)$ is a union of some countable family $\mathcal{F}_B$ of nowhere dense subsets in $\pi_B X$ (since LB is true!). But then $\cup\mathcal{N}_B \subseteq \pi_B^{-1}(\pi_B(\cup\mathcal{N}_B)) = \pi_B^{-1}(\cup\mathcal{F}_B) \subseteq \cup\{\pi_B^{-1}F : F \in \mathcal{F}_B\}$. It follows finally: $\cup\mathcal{N} \subseteq \cup\{\cup\{\pi_B^{-1}F : F \in \mathcal{F}_B\} : B \in \mathcal{B}\}$. $\square$

Let have a look at the function $Cov_{\aleph_0}$.

2.3 Proposition. (a) *If $m^{\aleph_0} > \mathfrak{C}$, then $Cov_{\aleph_0}(m) = m^{\aleph_0}$;*
(b) *if $cf(m) = \omega$, then $Cov_{\aleph_0}(m) > m$;*
(c) *if $cf(m) > \omega$, then $Cov_{\aleph_0}(m) = cf(m) \cdot Sup\{Cov_{\aleph_0}(n) : n \in m\}$.*

Proof: (a) It is clear that $[m]^{\leq\omega} = \cup\{[B]^{\leq\omega} : B \in \mathcal{B}\}$ for every covering base $\mathcal{B}$, hence, $|[m]^{\leq\omega}| = |\mathcal{B}| \cdot \mathfrak{C}$ and as $m^{\aleph_0} > \mathfrak{C}$ so $m^{\aleph_0} = |\mathcal{B}|$. Finally, $m^{\aleph_0} = Cov_{\aleph_0}(m)$.

(b) Let $\mathcal{B}$ be a covering base of power m. By our hypothesis $m = \Sigma\{n_i : i \in \omega\}$. Thus let $n_i < n_{i+1}$ for every $i < \omega$. So, $\mathcal{B} = \cup\{\mathcal{B}_i : i \in \omega\}$, where $|\mathcal{B}_i| = n_i$. For every $i < \omega$ we have $n_{i+1} > n_i = |\mathcal{B}_i|$, so there exists some $b_{i+1} \in n_{i+1} \setminus \cup\mathcal{B}_i$. We see that the set $\{b_{i+1} : i \in \omega\}$ is not covered by any element of $\mathcal{B}$.

(c) It is evident, that on one hand $Cov_{\aleph_0}(m) \leq cf(m) \cdot Sup\{Cov_{\aleph_0}(n) : n \in m\}$ and on another hand, $Cov_{\aleph_0}(m) \geq Cov_{\aleph_0}(n)$ if $m \geq n$, hence $Cov_{\aleph_0}(m) \geq cf(m) \cdot Sup\{Cov_{\aleph_0}(n) : n \in m\}$. $\square$

2.4 Corollary from Proposition 2.3. $Cov_{\aleph_0}(\aleph_n) = \aleph_n$ *for every $n < \omega$ and* $Cov_{\aleph_0}(\aleph_\omega) > \aleph_\omega$.

2.5 Proposition. *In any generic extension by means of a partially ordered* (p.o.) *subset $\mathcal{P}$ with countable Suslin number, i.e. $c(\mathcal{P}) \leq \aleph_0$ (i.e. with c.c.c.) the function $Cov_{\aleph_0}$ is the same as in the ground model.*

Proof: Let $\mathfrak{M}$ be a ground model, $\mathfrak{N}$ be its generic extension by means of a p.o. set $\mathcal{P}$ with $c(\mathcal{P}) \leq \aleph_0$. Each countable subset A of a cardinal m in $\mathfrak{N}$ has a name $\dot{A}$ in $\mathfrak{M}$, which is a subset of $\omega \times m \times \mathcal{P}$. As $c(\mathcal{P}) \leq \aleph_0$ so for every $n \in \omega$ a family $\{(\alpha, p) : (n, \alpha, p) \in \dot{A}\}$ can be assumed to be countable. Let $K(\dot{A}) = \{\alpha :$ there exist $n \in \omega, p \in \mathcal{P}$ such that $(n, \alpha, p) \in \dot{A}\}$, then $K(\dot{A}) \in \mathfrak{M}$ and $K(\dot{A})$ is a countable subset of $\mathfrak{M}$. As it is easy to see, $A \subseteq K(\dot{A})$ in $\mathfrak{N}$. This implies that every covering base of $[m]^{\leq\omega}$ from $\mathfrak{M}$ is a covering base of $[m]^{\leq\omega}$ in $\mathfrak{N}$ as well. Now the conclusion of the proposition is evident. □

2.6 Corollary from Proposition 2.5. *Let $\mathfrak{N}$ be a generic extension by means of p.o. set with c.c.c. of a model $\mathfrak{M}$ in which GCH is valid. Then in $\mathfrak{N}$*

$$Cov_{\aleph_0}(m) = \begin{cases} m, & \text{if } cf(m) > \omega \\ m^+, & \text{if } cf(m) = \omega. \end{cases}$$

3. Answer to Shapiro's question

3.1 Theorem. (a) *Under LB for every $\tau < \aleph_\omega$ the assertion $(*)$ is fulfilled;*

(b) *under* $[\mathrm{LB} + \mathfrak{c} = \aleph_{\omega+1}]$ *the assertion $(*)$ is fulfilled;*

(c) *if GCH is valid in a ground model and LB is valid in the generic extension $\mathfrak{N}$ (by means of a p.o. set with c.c.c.) then in $\mathfrak{N}$:*

(c1) *the assertion $(*)$ is fulfilled for τ iff $cf(\tau) > \omega$;*

(c2) *the assertion $(**)$ is fulfilled for every τ ($< \mathfrak{C}$, of course).*

Proof: (a),(b) and (c2) follow from Theorem 2.2 and Corollaries 2.4, 2.6. Let us prove (c1). The first part of (c1) is clear, so we have to prove only its second part, i.e. if $cf(\tau) = \omega$, then $(*)$ is not fulfilled. A proof for every countably cofinal cardinal is the same as for $\aleph_\omega$, what we will do.

For every $n \in \omega$ let $X_n = \omega_n$, with all sets X_n disjoint. Let $X = \Sigma\{X_n : n \in \omega\}$. In the ground model $\mathfrak{M}$ $|[X]^{\leq\omega}| = \aleph_{\omega+1}$. Let us denote the set of all countable subsets of X by Y. As the extension $\mathfrak{M} \to \mathfrak{N}$ has been made by means of p.o. set with c.c.c. so $|Y| = \aleph_{\omega+1}$ in $\mathfrak{N}$ (let us note that Y is not the set of all countable subsets of X in $\mathfrak{N}$).

Let us consider in $\mathfrak{N}$ the following family $\mathcal{Y}$ of nowhere dense subsets of 2^X. For every $A \in Y$ let N_A be $\pi_A^{-1}\{\bar{0}\}$, where $\bar{0}$ is the function: $A \to 0$. The family $\mathcal{Y} = \{N_A : A \in \mathcal{Y}\}$ of nowhere dense subsets of 2^X has in $\mathfrak{N}$ the power $\aleph_{\omega+1}$, hence $|\mathcal{Y}| = \aleph_{\omega+1} < \mathfrak{C}$. Let us show that in $\mathfrak{N}$ there exists no family $\mathcal{E}$ of cardinality not greater than $\aleph_\omega$ of nowhere dense subsets and such that $\cup\mathcal{E} = \cup\mathcal{Y}$. Assume the contrary and let $\mathcal{E}$ be a such family. For every $E \in \mathcal{E}$ let A_E be a such countable subset of X that $E \subseteq \pi_{A_E}^{-1}\pi_{A_E}E$. So we can think that every $E \in \mathcal{E}$ generates a pair: a countable subset $A_E \subseteq X$ and a nowhere dense subset $\pi_{A_E}E \subseteq 2^{A_E}$. Let us analyze first the family $\{A_E : E \in \mathcal{E}\}$. □

3.2 Proposition. *In $\mathfrak{N}$ for every family $\mathcal{A}$ of countable subsets of X such that $|\mathcal{A}| \leq \aleph_\omega$ there exists $B \in Y$ such that $|B \cap A| < \aleph_0$ for every $A \in \mathcal{A}$.*

Proof: Let us come back to $\mathfrak{M}$. A name $\dot{\mathcal{A}}$ is a some set of triples $(\alpha, p, \dot{A})$, where $\alpha \in \omega_\omega$, $p \in \mathcal{P}$, and $\dot{A}$ is a name of countable subset of X. As $c(\mathcal{P}) \leq \aleph_0$ so for every $\alpha \in \omega_\omega$ the set of all triples $(\alpha, p, \dot{A}) \in \mathcal{A}$ is countable. And what is a name $\dot{A}$? It is a set of triples (n, β, p), where $n \in \omega$, $\beta \in X$, $p \in \mathcal{P}$, as $c(\mathcal{P}) \leq \aleph_0$ for every $n \in \omega$ the set of all triples $(n, \beta, p) \in \dot{A}$ is countable. Let $K(\dot{A}) = \{\beta :$ there exists $(n, \beta, p) \in \dot{A}\}$. Then $K(\dot{A})$ is the countable subset of X. Let us consider the family $\mathcal{K} = \{K(\dot{A})$: there exists some $(\alpha, p, \dot{A}) \in \dot{\mathcal{A}}\}$. It is evident that $|\mathcal{K}| = \aleph_\omega$ and as it is easy to see that $\mathcal{A} \subseteq \mathcal{K}$ in $\mathfrak{N}$. As $|\mathcal{K}| = \aleph_\omega$, $\mathcal{K}$ can be represented as a union $\cup\{\mathcal{K}_n : n \in \omega\}$ of sets $\mathcal{K}_n$ of the power $\aleph_n$. It is evident that for every $n \in \omega$ there exists some $b_{n+1} \in X_{n+1} \setminus \cup\mathcal{K}_n$ and $B = \{b_{n+1} : n \in \omega\}$ is the desired subset. □

Let us come in $\mathfrak{N}$.

3.3 Proposition [LB]. *Let $\mathcal{A}$ be a family of countable subsets of X, $|\mathcal{A}| < \mathfrak{C}$ and B be a such countable subset of X, that $|B \cap A| < \aleph_0$ for every $A \in \mathcal{A}$. Let a nowhere dense subset $N \subseteq 2^B$ and for every $A \in \mathcal{A}$ a nowhere dense subset $N_A \subseteq 2^A$ be given. Then $\pi_B^{-1} N \not\subset \cup\{\pi_a^{-1} N_A : A \in \mathcal{A}\}$.*

Proof: Let N be a one point set θ. Then $\pi_B^{-1}\{\theta\} = \{\theta\} \times 2^{X \setminus B}$. Let us denote this subspace by Z for brevity. As $|X| \leq \mathfrak{C}$ so Z is a separable compactum, hence, under LB it is not represented as a union of less than $\mathfrak{C}$ of nowhere dense subsets ([2]). Now we have only to prove that every subset $\pi_A^{-1} N_A$ is nowhere dense in Z, i.e. $Z \cap \pi_A^{-1} N_A$ is nowhere dense in Z.

Let V be any open nonempty subset of Z, then we can assume that $V = \{z \in 2^X : z \supset \varphi\}$ where φ is some finite partial function from X into $\{0,1\}$ and $\varphi/B = \theta/B$. But $|A \cap B| < \aleph_0$ so φ can be extended to a function ψ such that $dom\psi \supseteq A \cap B$. As $\pi_A^{-1} N_A$ is nowhere dense in 2^X so that there exists a finite function f extending ψ and such that $f/B = \theta/B, F \cap \pi_A^{-1} N_A = \emptyset$ where $F = \{z \in 2^X : z \supset f\}$. But $F \subset V, F \cap Z \neq \emptyset$ and $F \cap Z \cap \pi_A^{-1} N_A = \emptyset$. It has been proved that $\pi_A^{-1} N_A$ is nowhere dense in Z.

The theorem has been proved. □

3.4 Remark. *In proofs it is essential that considered spaces are Tychonoff. Is Theorem* 3.1 *true for regular spaces?*

3.5 Remark. *It is known that under MA the ideal of subsets of reals of Lebesgue measure* 0 *is* $(< \mathfrak{c})$*-additive. How is "Shapiro's" question to be answered for Lebesgue measure (or some others measures)?*

Acknowledgment. I would like to express my gratitude to V.V. Tkachuk for the help in the preparation of this article.

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(*Received* December 14, 1994)