A remark on the tightness of products

Oleg Okunev

Abstract. We observe the existence of a σ-compact, separable topological group G and a countable topological group H such that the tightness of G is countable, but the tightness of $G \times H$ is equal to $\mathfrak{c}$.

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All topological spaces below are assumed Tychonoff (= completely regular Hausdorff). The *tightness* $t(X)$ of a space X is defined as the minimal infinite cardinal τ such that for any $A \subset X$, the closure of A in X coincides with the union of closures of all subsets B of A with $|B| \leq \tau$.

In 1972, Malykhin proved [Mal] that if X is compact, then for any space Y, $t(X \times Y) = t(X)t(Y)$. The question whether the same is true under weaker assumptions on X was answered by Arhangel'skii in [Arh1] where he showed that multiplication by a first countable space does not raise the tightness, but the product of a countably tight space and a countable space may have uncountable tightness. However, the question whether the equality $t(X \times Y) = t(X)t(Y)$ holds if both X and Y (or even $X \times Y$) have good compactness properties (Lindelöf, σ-compact or Lindelöf Σ-spaces) remained open. Shakhmatov [Sha] attributed the question about Lindelöf Σ-spaces to Kombarov [Kom], and describes (quite correctly) the question in the σ-compact case as "a kind of folklore question raised from time to time at the Moscow topological seminar".

In [Tod], Todorčević constructed two countably tight, σ-compact topological groups whose product has uncountable tightness, thus answering the above questions in the negative. In this paper we improve slightly this result by making the second factor countable. Our example involves no advanced combinatorics (see [OT] for a straightforward construction of the underlying space X below). It should be noted however that our groups are not Frechét-Urysohn, nor compactly generated as in Todorčević's example.

Theorem 1. *There exist a separable, σ-compact topological group G and a countable topological group H such that $t(G^\omega) = \omega$ and $t(G \times H) = \mathfrak{c}$.*

Proof: Given two spaces X and Y, we denote by $C_p(X, Y)$ the space of all continuous functions from X to Y equipped with the topology of pointwise convergence (see [Arh2]). We will be particularly interested in the case $Y = 2 = \{0, 1\}$. Note that for any family of spaces $\{ X_\gamma : \gamma \in \Gamma \}$, there is a standard homeomorphism between $C_p(\bigoplus\{ X_\gamma : \gamma \in \Gamma \}, Z)$ and $\prod\{ C_p(X_\gamma, Z) : \gamma \in \Gamma \}$.

Let $\mathbf{C} = 2^\omega$ be the Cantor discontinuum; for a subset A of $\mathbf{C}$, we denote by $\mathbf{C}_{(A)}$ the space obtained by retaining the original topology at all points of A and declaring all points of $\mathbf{C} \setminus A$ isolated. It follows easily from Theorems 1 and 2 in [Law] (see also [OT]) that there is a subset A of $\mathbf{C}$ such that the set $B = \mathbf{C} \setminus A$ has cardinality $\mathfrak{c}$ and all finite powers of the space $X = \mathbf{C}_{(A)}$ is Lindelöf.

Let $i\colon X \to \mathbf{C}$ be the natural bijection; we denote $B_0 = i^{-1}(B)$. B_0 is an uncountable, discrete set in X.

Let $\mathcal{C}$ be the set of all clopen sets in $\mathbf{C}$. Clearly, $\mathcal{C}$ is countable and for any finite disjoint subsets F_1 and F_2 of $\mathbf{C}$, $F_1 \subset U$ and $F_2 \cap U = \emptyset$ for some $U \in \mathcal{C}$.

Assign to each pair of a finite set $F \subset B$ and $U \in \mathcal{C}$ such that $F \cap U = \emptyset$ the function $f_{F,U}\colon X \oplus B \to 2$ defined by the rule

$$f_{F,U}(x) = \begin{cases} 0 & \text{if } x \in i^{-1}(F) \text{ or } x \in i^{-1}(U) \text{ or } x \in B \setminus U, \\ 1 & \text{otherwise} \end{cases}$$

and put $S = \{ f_{F,U} : F \text{ is a finite subset of } B,\ U \in \mathcal{C} \text{ and } F \cap U = \emptyset \}$. Clearly, $S \subset C_p(X \oplus B, 2)$.

Claim 1. *The zero function* 0 *is a limit point* (*in* $C_p(X \oplus B, 2)$) *for* S, *but not a limit point for any subset of* S *whose cardinality is less than* $\mathfrak{c}$.

Indeed, a generic neighborhood of 0 in $C_p(X \oplus B, 2)$ is of the form $O(K) = \{ f \in C_p(X \oplus B, 2) : f|K = 0 \}$ where K is a finite set in $X \oplus B$. Let $K_1 = K \cap B_0$, $K_2 = K \cap (X \setminus B_0)$ and $K_3 = K \cap B$. Put $F = (i(K_1) \cup K_3)$, and find a set $U \in \mathcal{C}$ so that $i(K_2) \subset U$ and $U \cap F = \emptyset$; then $f_{F,U}$ is in $S \cap O(K)$ (note that always $f_{F,U} \neq 0$). Thus, 0 is a limit point for S.

Now suppose S_1 is a subset of S of cardinality less than $\mathfrak{c}$. Then the set $M = \bigcup\{ F : f_{F,U} \in S_1 \text{ for some } U \in \mathcal{C} \}$ has cardinality less than $\mathfrak{c}$; pick a point $x_0 \in B \setminus M$ and put $x_1 = i^{-1}(x_0)$. Now for any $f = f_{F,U} \in M$, we have either $f(x_0) = 1$ or $f(x_1) = 1$, and $O(\{x_0, x_1\})$ is a neighborhood of 0 disjoint with S_1.

Claim 2. *The set* S *is* σ*-compact.*

We have $S = \bigcup\{ \{ f_{F,U} : |F| \le n,\ F \cap U = \emptyset \} : U \in \mathcal{C}, n \in \omega \}$, and each set $S_{n,U} = \{ f_{F,U} : |F| \le n, F \cap U = \emptyset \}$ is compact, because it is closed in 2^X.

We have $S \subset C_p(X \oplus B, 2) \simeq C_p(X, 2) \times C_p(B, 2)$; let S_1 and S_2 be the projections of S on the factors $C_p(X, 2)$ and $C_p(B, 2)$, L a dense countable subset of $C_p(X, 2)$ ($C_p(X, 2)$ is separable, because X admits a continuous bijection to $\mathbf{C}$, see Theorem I.1.5 in [Arh2]) and G and H be the subgroups of $C_p(X, 2)$ and $C_p(B, 2)$ generated by $S_1 \cup L$ and S_2. Clearly, H is countable, and G is separable and σ-compact. Furthermore, $G \subset C_p(X, 2)$, and since all finite powers of X are Lindelöf, so are all finite powers of $X \times \omega$, and the tightness of $C_p(X \times \omega, 2) = C_p(X, 2)^\omega$ is countable by a theorem of Arhangel'skii and Pytkeev (see [Arh2, II.1.1]), so the tightness of G^ω is countable. The product of G and H contains $S \cup \{0\}$, so by Claim 1, the tightness of $G \times H$ is equal to $\mathfrak{c}$. □

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The University of Aizu, Fukushima 965-80, Japan

E-mail: o-okunev@u-aizu.ac.jp

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