A fixed point theorem for a multivalued non-self mapping

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Abstract. We prove a fixed point theorem for a multivalued non-self mapping in a metrically convex complete metric space. This result generalizes Theorem 1 of Itoh [2].

Keywords: multivalued non-self mapping, metrically convex metric space

Classification: 47H10, 54H25

Let $CB(X)$ denote the family of all nonempty closed bounded subsets of X, H the Hausdorff metric on X, and for any set A, $D(x, A)$ denotes the distance from x to A.

Lemma 1 ([3]). *If $A, B \in CB(X)$ and $x \in A$, then for any positive number ε there exists a $y \in B$ such that $d(x, y) \leq H(A, B) + \varepsilon$.*

Lemma 2 ([1]). *Let K be a nonempty closed subset of a complete metrically convex metric space (X, d). Then, for any $x \in K$, $y \notin K$, there exists a point $z \in \partial K$ such that $d(x, z) + d(z, y) = d(x, y)$.*

Let $F : X \to CB(X)$. We shall be interested in the following contractive definition. For each x, y in X,

$$(1) \quad H(Fx, Fy) \leq \\ \leq \alpha d(x, y) + \beta \max\{D(x, Fx), D(y, Fy)\} + \gamma[D(x, Fy) + D(y, Fy)],$$

where $\alpha, \beta, \gamma \geq 0$ and such that

$$s := \left(\frac{1 + \alpha + \gamma}{1 - \beta - \gamma}\right)\left(\frac{\alpha + \beta + \gamma}{1 - \gamma}\right) < 1.$$

Theorem 1. *Let (X, d) be a complete metrically convex metric space, K a nonempty closed subset of X. Let $F : X \to CB(X)$ satisfy* (1). *If for each $x \in \partial K$, $Fx \subset K$, then there exists a $z \in K$ such that $z \in Fz$.*

Proof: We shall construct two sequences $\{x_n\}$ and $\{x'_n\}$ as follows. Let $x_0 \in K$. Let $x'_1 \in Tx_0$. If $x'_1 \in K$, set $x_1 = x'_1$. If $x'_1 \notin K$, choose x_1 so that $d(x_0, x_1) + d(x_1, x'_1) = d(x_0, x'_1)$. Pick $x'_2 \in Fx_1$ so that $d(x_1, x'_2) \leq H(Fx_0, Fx_1) + (1 - \beta - \gamma)\varepsilon$, with $\varepsilon = s$. If $x'_2 \in K$, set $x_2 = x'_2$. If $x'_2 \notin K$, since X is metrically convex, choose x_2 so that $d(x_1, x_2) + d(x_2, x'_2) = d(x_1, x'_2)$. Continuing in this way we obtain two sequences $\{x_n\}$, $\{x'_n\}$ satisfying the following properties: $x'_{n+1} \in Fx_n$ and such that $d(x'_n, x'_{n+1}) \le H(Fx_{n-1}, Fx_n) + (1-\beta-\gamma)\varepsilon^n$; $x_{n+1} = x'_{n+1}$ if $x'_{n+1} \in K$; and, if $x'_{n+1} \notin K$, x_{n+1} is chosen to satisfy $d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$.

Define

$$P = \left\{x_n \in \{x_n\} : x_n = x'_n\right\}, \ Q = \left\{x_n \in \{x_n\} : x_n \ne x'_n\right\}.$$

Three cases now arise.

Case 1. $x_n, x_{n+1} \in P$. From (1),

$$\begin{aligned} d(x_n, x_{n+1}) &\le H(Fx_{n-1}, Fx_n) + (1-\beta-\gamma)\varepsilon^n \\ &\le \alpha d(x_{n-1}, x_n) + \beta \max\{D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \\ &\qquad + \gamma[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})] + (1-\beta-\gamma)\varepsilon^n \\ &\le \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &\qquad + \gamma d(x_{n-1}, x_{n+1}) + (1-\beta-\gamma)\varepsilon^n \\ &\le \max\left\{\frac{(\alpha+\beta+\gamma)d(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\gamma},\right. \\ &\qquad \left.\frac{(\alpha+\gamma)d(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\beta-\gamma}\right\} \\ &\le \max\left\{\frac{\alpha+\beta+\gamma}{1-\gamma}, \frac{\alpha+\gamma}{1-\beta-\gamma}\right\} d(x_{n-1}, x_n) + \varepsilon^n \\ &= kd(x_{n-1}, x_n) + \varepsilon^n, \end{aligned} \tag{2}$$

where

$$k := \frac{\alpha+\beta+\gamma}{1-\gamma}.$$

Case 2. $x_n \in P$, $x_{n+1} \in Q$. From (1),

$$\begin{aligned} d(x_n, x_{n+1}) \le d(x_n, x'_{n+1}) &\le H(Fx_{n-1}, Fx_n) + (1-\beta-\gamma)\varepsilon^n \\ &\le \alpha d(x_{n-1}, x_n) + \beta \max\{D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \\ &\qquad + \gamma[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})] + (1-\beta-\gamma)\varepsilon^n \\ &\le \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, x_n), d(x_n, x'_{n+1})\} \\ &\qquad + \gamma d(x_{n-1}, x'_{n+1}) + (1-\beta-\gamma)\varepsilon^n. \end{aligned}$$

If the maximum of the coefficient of β is $d(x_{n-1}, x_n)$, then we have

$$d(x_n, x'_{n+1}) \le \frac{(\alpha+\beta+\gamma)d(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\gamma}.$$

If the maximum of the coefficient of β is $d(x_n, x'_{n+1})$, then we have

$$d(x_n, x'_{n+1}) \le \frac{(\alpha+\gamma)d(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\beta-\gamma}.$$

Therefore, in all cases,

$$d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n) + \varepsilon^n. \tag{3}$$

Case 3. $x_n \in Q$, $x_{n+1} \in P$. Then, using (1) and Case 2,

$$\begin{aligned}
d(x_n, x_{n+1}) &\le d(x_n, x'_n) + d(x'_n, x_{n+1}) \\
&\le d(x_n, x'_n) + H(Fx_{n-1}, Fx_n) + (1-\beta-\gamma)\varepsilon^n \\
&\le d(x_{n-1}, x'_n) + \alpha d(x_{n-1}, x_n) \\
&\qquad + \beta \max\{D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \\
&\qquad + \gamma[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})] + (1-\beta-\gamma)\varepsilon^n \\
&\le d(x_{n-1}, x'_n) + \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, x'_n), d(x_n, x_{n+1})\} \\
&\qquad + \gamma[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)] + (1-\beta-\gamma)\varepsilon^n.
\end{aligned}$$

Note that

$$\begin{aligned}
d(x_{n-1}, x_{n+1}) + d(x_n, x'_n) &\le d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x'_n) \\
&= d(x_{n-1}, x'_n) + d(x_n, x_{n+1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
d(x_n, x_{n+1}) &\le \left\{ \frac{(1+\alpha+\beta+\gamma)d(x_{n-1}, x'_n) + (1-\beta-\gamma)\varepsilon^n}{1-\gamma} \right. \\
&\qquad \left. \frac{(1+\alpha+\gamma)d(x_{n-1}, x'_n) + (1-\beta-\gamma)\varepsilon^n}{1-\beta-\gamma} \right\} \\
&\le \max\left\{ \frac{1+\alpha+\beta+\gamma}{1-\gamma}, \frac{1+\alpha+\gamma}{1-\beta-\gamma} \right\} d(x_{n-1}, x'_n) + \varepsilon^n \\
&\le \left(\frac{1+\alpha+\gamma}{1-\beta-\gamma} \right) d(x_{n-1}, x'_n) + \varepsilon^n \\
&\le sd(x_{n-2}, x_{n-1}) + \frac{(1+\alpha+\gamma)\varepsilon^{n-1}}{1-\beta-\gamma} + \varepsilon^n.
\end{aligned}$$

Using (2)–(4), it can be shown by induction that

$$d(x_{2n}, x_{2n+1}) \le s^n \left(\delta + \frac{3n}{1-\beta-\gamma} \right)$$

and that

$$d(x_{2n+1}, x_{2n+2}) \le s^{(2n+1)/2}\left(\delta + \frac{3n+1}{1-\beta-\gamma}\right).$$

Then, for any $m > n$,

$$d(x_m, x_n) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \delta \sum_{i=n}^{m-1} s^{i/2} + \frac{1}{1-\beta-\gamma}\sum_{i=n}^{m-1} s^{i/2}(3i+1),$$

and $\{x_n\}$ is Cauchy, hence convergent. Call the limit z. From the way in which the $\{x_n\}$ were chosen, there exists an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = x'_{n_k}$.

$$\begin{aligned} D(x_{n_k}, Fz) &\le H(Fx_{n_i-1}, Fz) \\ &\le \alpha d(x_{n_k-1}, z) + \beta \max\{D(x_{n_k-1}, Fx_{n_k-1}), D(z, Fz)\} \\ &\qquad + \gamma[D(x_{n_k-1}, Fz) + D(z, Fx_{n_k-1})] \\ &\le \alpha d(x_{n_k-1}, z) + \beta \max\{d(x_{n_k-1}, x_{n_k}), D(z, Fz)\} \\ &\qquad + \gamma[D(x_{n_k-1}, Fz) + d(z, x_{n_k})]. \end{aligned}$$

Taking the limit as $k \to \infty$ yields $D(z, Fz) \le (\beta+\gamma)D(z, Fz)$, which implies that $z \in Fz$. □

References

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(*Received* September 2, 1994, *revised* November 24, 1995)