On extension of functors to the Kleisli category of some weakly normal monads

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Abstract. The problem of extension of normal functors to the Kleisli categories of the inclusion hyperspace monad and its submonads is considered. Some negative results are obtained.

Keywords: monad, Kleisli category, weakly normal functor, inclusion hyperspace, superextension

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0. Introduction and denotation

The paper is devoted to the general problem of extension of functors onto the Kleisli category of a monad. We restrict ourselves to the case of weakly normal functors and monads in the category of compacta (see the definitions below).

A triple $\mathbb{T} = (T, \eta, \mu)$ is said to be a monad on the category $\mathcal{C}$ ([1], [2]) if T is endofunctor acting in $\mathcal{C}$ and $\eta : 1_{\mathcal{C}} \to T$, $\mu : T^2 \to T$ are natural transformations satisfying $\mu \circ \eta T = \mu \circ T\eta = 1_T$, $\mu \circ \mu T = \mu \circ T\mu$.

Recall the notion of Kleisli category $\mathcal{C}_{\mathbb{T}}$ of the monad $\mathbb{T}$ (see, e.g., [2]). The objects of both categories $\mathcal{C}$ and $\mathcal{C}_{\mathbb{T}}$ are the same, and $\mathcal{C}_{\mathbb{T}}(X,Y) = \mathcal{C}(X,TY)$. The composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$ and $g \in \mathcal{C}_{\mathbb{T}}(Y,Z)$ is defined by the formula $g * f = \mu Z \circ Tg \circ f$.

Denote by $I : \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ the functor being identity on objects and satisfying $If = \eta Y \circ f \in \mathcal{C}_{\mathbb{T}}(X,Y)$, $f \in \mathcal{C}(X,Y)$.

Definition. A functor $\overline{F} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ is called an extension of the functor $F : \mathcal{C} \to \mathcal{C}$ to the Kleisli category $\mathcal{C}_{\mathbb{T}}$ of the monad $\mathbb{T}$ if the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I} \\ \mathcal{C}_{\mathbb{T}} & \xrightarrow[\overline{F}]{} & \mathcal{C}_{\mathbb{T}}. \end{array}$$

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Proposition 1 (see J. Vinárek [3]). *There exists the bijective correspondence between the class of all extensions of the functor F onto the category $\mathcal{C}_{\mathbb{T}}$ and the set of all natural transformations $\xi : FT \to TF$ satisfying:*

(1) $\xi \circ F\eta = \eta F$,

(2) $\mu F \circ T\xi \circ \xi T = \xi \circ F\mu$.

In the sequel we shall consider only the category *Comp* compacta (= compact Hausdorff spaces) and their continuous maps.

Recall the constructions of the inclusion hyperspace monad $\mathbb{G}$ and its submonads ([2]). For a compact X we denote by $\exp X$ the set of nonempty closed subsets of X endowed with Vietoris topology. A base of this topology consists of the sets of the form

$$\langle U_1, \ldots, U_n \rangle = \Big\{ A \in \exp X \mid A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \varnothing \ \text{ for each } \ i = 1, \ldots, n \Big\},$$

where $U_1, \ldots, U_n$ are open in X. The set $\mathcal{A} \in \exp^2 X$ is said to be an inclusion hyperspace if $[A \in \mathcal{A}, A \subset B \Rightarrow B \in \mathcal{A}]$. Denote by GX the set of inclusion hyperspaces with the inherited from $\exp^2 X$ topology.

For a map $f : X \to Y$ the map $Gf : GX \to GY$ is defined by the formula $Gf(\mathcal{A}) = \{B \in \exp Y \mid B \supset f(A) \text{ for some } A \in \mathcal{A}\}$, $\mathcal{A} \in GX$. Obviously, we obtain a functor $G : Comp \to Comp$ (the inclusion hyperspace functor).

The functor G determines a monad $\mathbb{G} = (G, \eta, \mu)$ where $\eta X(x) = \{A \in \exp X \mid x \in A\}$, $\mu X = \cup\{\cap \mathcal{A} \mid \mathcal{A} \in \mathfrak{A}\}$, $x \in X$, $\mathfrak{A} \in G^2 X$ ([4]).

By $N_k X$, $k \geq 2$, we denote the subspace of GX consisting of all k-linked systems of closed subsets of X (recall that the system is called k-linked if the intersection of every its k-element subsystem is nonempty), and by $N_\infty X$ we denote the space $\bigcap_{k=1}^{\infty} N_k X$. Denote also by λX the subspace of GX consisting of all maximal with respect to inclusion 2-linked systems of closed subsets of X. The space λX is called the superextension of X ([5]). The maps $N_k f$, $N_\infty f$ and λf are defined by the same formulae as Gf.

The functors N_k, N_∞ and λ with the above-mentioned natural transformations η and μ form the submonads $\mathbb{N}_k$, $\mathbb{N}_\infty$ and $\mathbb{L}$ of monad $\mathbb{G}$ respectively.

The following definition is due to E. Shchepin [6]. A functor $F : Comp \to Comp$ is called weakly normal (normal) if it is continuous, monomorphic, epimorphic, preserves weight, intersections, (preimages), singletons and empty set. The monad is said to be (weakly) normal if so is its functorial part.

Note that monads $\mathbb{G}$, $\mathbb{N}_k$, $\mathbb{N}_\infty$ and $\mathbb{L}$ are weakly normal ([2]).

Denote by $t(x_1, \ldots, x_{k+1})$ the k-linked system $\{A \in \exp X \mid |A \cap \{x, \ldots, x_{k+1}\}| \geq k\}$, $x_i \in X$ are distinct, $k \in \mathbb{N}$.

Recall also that for a weakly normal functor the support $\mathrm{supp}(a)$ of a point $a \in FX$ is defined ([2]): $\mathrm{supp}(a) = \cap\{A \in \exp X \mid FA \ni a\}$ (here FA is identified with $Fi(A)$ where the map $i : A \to X$ is the natural embedding). We say that the degree of functor F is $\leq n$ (briefly, $\deg F \leq n$) if $|\mathrm{supp}(a)| \leq n$ for every

$a \in FX$, $X \in Comp$. If moreover $\deg F \nleq n-1$, we say that the degree F equals n, $\deg F = n$.

1. Main results

Now we aim to prove the following theorem.

Theorem 1. *No weakly normal functor of finite degree $n > 1$ extends onto the Kleisli category of the k-linked system monad $\mathbb{N}_k$, $k \geq 2$.*

Note that it is shown in [7] that the functor $(-)^2$ fails to extend onto the Kleisli category of the superextension monad $\mathbb{L}$. Here we generalize the method used in [7].

Let F be a weakly normal functor satisfying

$$(A_k) \qquad \begin{gathered} \exists\, n,\ \exists\, a \in F\{1,\dots,n\} \setminus \{1,\dots,n\} \ \text{ such that for the retraction} \\ r : \{1,\dots,n+k-1\} \to \{1,\dots,n\},\ r(\{n+1,\dots,n+k-1\}) = n \\ \text{the following holds: } \ |(Fr)^{-1}(a)| = k. \end{gathered}$$

The condition (A_k) means that only copies of a in $F\{1,\dots,n-1,l\}$, $l = n,\dots,n+k-1$ belong to $(Fr)^{-1}(a)$.

It will be convenient to introduce some notation. Let X be a finite compactum. For every ordered pair (x,y), $x \neq y$, $x,y \in X$ the point $a(x,y) \in F(X \sqcup \{1,\dots,n\})$ is defined by the formula: $a(x,y) = Ff_{xy}(a)$, where $f_{xy} : \{1,\dots,n\} \to \{x,y,1,\dots,n-2\} \subset X \sqcup \{1,\dots,n\}$ is the inclusion, $f_{xy}(i) = i$, $i \leq n-2$, $f_{xy}(n-1) = x$, $f_{xy}(n) = y$. Observe that $a(x,y)$ is a copy of a in $F\{x,y,1,\dots,n-2\}$.

Proposition 2. *Let F be a weakly normal functor satisfying (A_k), $k \geq 2$. Then F does not extend onto the Kleisli category of the monad $\mathbb{N}_k$.*

Proof: To the contrary, assume that $F : Comp \to Comp$ satisfies (A_k) and $\xi : FN_k \to N_kF$ is a natural transformation such that the conditions (1) and (2) of Proposition 1 hold.

Denote by X and Y disjoint discrete sets $\{x_1,\dots,x_{k+1}\}$ and $\{y_1,\dots,y_{k+1}\}$ respectively. Let Z be a space $X \sqcup Y \sqcup \{1,\dots,n-2\}$. We shall write a_{ij} instead of $a(x_i,y_i)$ $i,j = 1,\dots,k+1$.

Claim 1. For the retraction $f : Z \to Z \setminus (X \setminus \{x_i,x_j\})$, $f(X \setminus \{x_i,x_j\}) = x_i$ $(i \neq j)$ the following holds: $(Ff)^{-1}(a_{im}) = \{a_{lm} \,|\, l \neq j\}$, $m = 1,\dots,k+1$.

The equalities $f \,|\, \{x_l, y_m, 1,\dots,n-2\} = f_{x_iy_m} \circ f^{-1}_{x_ly_m}$ and $F(f_{x_iy_m} \circ f^{-1}_{x_ly_m})(a_{lm}) = a_{im}$ and the condition (A_k) imply this claim.

Let $\overline{x} = t(x_1,\dots x_{k+1}) \in N_kZ$, $\overline{y} = t(y_1,\dots,y_{k+1}) \in N_kZ$ and consider the k-linked system $\mathcal{A} = \xi Z(a(\overline{x},\overline{y}))$.

Let $\mathcal{H}$ be a class of bijections from Z into Z transforming the sets X and Y onto themselves and being identity on $\{1,\dots,n-2\}$.

Claim 2. Let $h \in \mathcal{H}$. Then

$$(B) \qquad N_k Fh(\mathcal{A}) = \mathcal{A};$$

if, moreover, $h(x_i) = x_j$ $(h(y_i) = y_j)$, $i \neq j$, then $Fh(a_{im}) = a_{jm}$ $(Fh(a_{mi}) = a_{mj})$, $m = 1, \dots k$.

Since $N_k h$ is the identity on $\{\overline{x}, \overline{y}\}$, we obtain that $N_k Fh(\mathcal{A}) = N_k Fh \circ \xi Z(a(\overline{x}, \overline{y})) = \xi Z \circ F N_k h(a(\overline{x}, \overline{y})) = \xi Z(a(\overline{x}, \overline{y})) = \mathcal{A}$. The proof of the rest is similar to that of the preceding claim, and we left it to the reader.

In the sequel the property (B) will be called the invariantness of $\mathcal{A}$.

Now let $g : Z \to Z \setminus \{x_2, \dots, x_k, y_2, \dots, y_k\}$ be a retraction such that $g(\{x_2, \dots, x_k\}) = x_1$, $g(\{y_2, \dots, y_k\}) = y_1$. Then $N_k g(\overline{x}) = \eta Z(x_1)$, $N_k g(\overline{y}) = \eta Z(y_1)$ and we conclude that $N_k Fg(\mathcal{A}) = \xi Z \circ F N_k g(a(\overline{x}, \overline{y})) = \xi Z \circ F \eta Z(a_{11}) = \eta F Z(a_{11})$, because $N_k g \,|\, \{\overline{x}, \overline{y}, 1, \dots, n-2\} = \eta Z \circ f_{x_1 y_1} \circ f_{\overline{xy}}^{-1}$ (here we identify $\eta Z(i)$ and i, $i \leq n-2$). Consequently, the preimage $(Fg)^{-1}(a_{11})$ belongs to $\mathcal{A}$ as $\mathcal{A}$ is an inclusion hyperspace. Applying Claim 1 we find that $\{a_{lm} \,|\, l, m = 1, \dots, k\} = (Fg)^{-1}(a_{11}) \in \mathcal{A}$. Since $\mathcal{A}$ is invariant, it contains the family of "squares" $\{a_{lm} \,|\, l \neq i, m \neq j\}$, $i, j = 1, \dots, k+1$.

The following fact is obtained by a similar manner. We left its proof to the reader.

Claim 3. The k-linked system $\xi Z(a(\overline{x}, \eta Z(y_m)))$ $\big(\xi Z(a(\eta Z(x_m), \overline{y}))\big)$ contains every element of the system $t(a_{1m}, \dots, a_{k+1m})$ $\big(t(a_{m1}, \dots, a_{mk+1})\big)$, $m = 1, \dots, k+1$.

(More exactly in this claim one can write "equals" instead of "contains every element" but it is superfluous for our aim.)

Now we have all tools to prove the main proposition.

Denote by $\mathfrak{M}$ and $\mathfrak{N}$ the k-linked (in $N_k Z$) systems $t(\overline{x}, \eta Z(x_1), \dots, \eta Z(x_k))$ and $t(\overline{y}, \eta Z(y_1), \dots, \eta Z(y_k))$ respectively, and let $\mathcal{R} = a(\mathfrak{M}, \mathfrak{N})$.

Since $\mu Z(\mathfrak{M}) = \overline{x}$ and $\mu Z(\mathfrak{N}) = \overline{y}$, we conclude that $\xi Z \circ F \mu Z(\mathcal{R}) = \xi Z(a(\overline{x}, \overline{y})) = \mathcal{A}$. Therefore the condition (2) (of Proposition 1) implies that $\mathcal{A} = \mu F Z \circ N_k \xi Z \circ \xi N_k Z(\mathcal{R})$.

Now $\xi N_k Z(\mathcal{R})$ contains the sets

$$\mathcal{K} = \Big\{a(\eta Z(x_l), \eta Z(y_m)) \,|\, m \leq k, l = 2, \dots, k\Big\} \cup \Big\{a(\overline{x}, \eta Z(y_m)) \,|\, m \leq k\Big\},$$

$$\mathcal{L} = \Big\{a(\eta Z(x_m), \eta Z(y_l)) \,|\, m \leq k, l = 2, \dots, k\Big\} \cup \Big\{a(\eta Z(x_m), \overline{y}) \,|\, m \leq k\Big\}.$$

Since $\xi Z(a(\eta Z(x_i), \eta Z(y_j))) = \xi Z \circ F \eta Z(a_{ij}) = \eta F Z(a_{ij})$, we obtain that

$$\xi Z(\mathcal{K}) = \Big\{\eta F Z(a_{lm}) \,|\, m \leq k, l = 2, \dots, k\Big\} \cup \Big\{\xi Z(a(\overline{x}, \eta Z(y_m))) \,|\, m \leq k\Big\}.$$

Applying Claim 3 we can prove that $\mathcal{A}$ contains the set

$$A = \Big\{a_{11}, \dots, a_{k1}\Big\} \cup \Big\{a_{lm} \,|\, l = 2, \dots, k+1; m = 2, \dots, k\Big\}.$$

Similarly, considering the set $\mathcal{L}$, one can easily conclude that

$$A_1 = \Big\{a_{11}, \dots, a_{1k}\Big\} \cup \Big\{a_{ml} \,|\, l = 2, \dots, k+1; m = 2, \dots, k\Big\} \in \mathcal{A}.$$

Let $h_i \in \mathcal{H}$ be defined by the formula $h_i(x_1) = x_i$, $h_i(x_i) = x_{k+1}$, $h_i(x_{k+1}) = x_1$, $h_i(y_i) = y_{k+1}$, $h_i(y_{k+1}) = y_i$ and $h_i(t) = t$ for other t. Let $A_i = h_i(A)$, $i = 2, \dots, k$. It is easy to verify that $\bigcap_{i=1}^{k} A_i = \varnothing$. This contradicts the invariantness of $\mathcal{A}$. $\square$

Proposition 2 immediately yields Theorem 1.

Remark that condition (A_k) is quite technical. For the defined in Proposition 2 retraction g it implies the following: $(Fg)^{-1}(a_{11}) \subset \{a_{lm} \,|\, l, m = 1, \dots, k\}$. In general case the preimage $(Fg)^{-1}(a_{11})$ can be too large.

Theorem 2. *Let F be a weakly normal functor of a finite degree $n > 1$. Then F does not extend onto Kleisli category of the superextension monad.*

Proof: Since $t(x_1, x_2, x_3) \in \lambda X$ for every distinct $x_1, x_2, x_3 \in X$, we can argue such as in the proof of Proposition 2 for $k = 2$. $\square$

2. Remarks and open problems

Remark that proving Theorems 1, 2 we have used the k-linkness and invariantness of the considered systems. In the case of the inclusion hyperspace monad it fails the k-linkness of systems; and for the monad $\mathbb{N}_\infty$ it fails the "invariantness" of the considered systems.

Problem 1. *Describe the class of normal functors (of finite degree) extending onto the Kleisli category $Comp_{\mathbb{G}}$ of the inclusion hyperspaces monad.*

Problem 2. *Describe the class of normal functors (of finite degree) extending onto the category $Comp_{\mathbb{N}_\infty}$.*

Recall that a monad $\mathbb{T} = (T, \eta, \mu)$ is called projective if there exists a natural transformation $\pi : T \to 1_{Comp}$ such that $\pi \circ \eta = 1$, $\pi \circ \pi T = \pi \circ \mu$ ([3]).

Theorem (see [2], [8]). *Let $\mathbb{T}$ be a normal monad. Then the functor* $\exp_3$ *has an extension to the category $Comp_{\mathbb{T}}$ if and only if $\mathbb{T}$ is projective.*

(Here $\exp_3 X = \{a \in \exp X \,|\, \text{ the degree } \deg(a) \text{ of } a \leq 3\}$.)

Recall also the construction of G-symmetric power functor $SP^n_G : Comp \to Comp$ ([2], [9]). Let $n \in \mathbb{N}$ and a subgroup G of the symmetric group S_n acts on X^n by permutations of coordinates. The orbit space $SP^n_G X$ of this action is called the G-symmetric power of X. Let $[x_1, \dots, x_n]_G$ be the orbit containing the point $(x_1, \dots, x_n) \in X^n$. For a map $f : X \to Y$ the map $SP^n_G f : SP^n_G X \to SP^n_G Y$ is defined by the formula: $SP^n_G f[x_1, \dots, x_n]_G = [f(x_1), \dots, f(x_n)]_G$. The obtained functors SP^n_G are normal ([9]).

Theorem above and the results of papers [10], [11] arouse the interest to the following question.

Problem 3 (M. Zarichnyi)**.** *Are there a non-projective normal monad $\mathbb{T}$ and a normal differing from SP^n_G functor of finite degree F such that F extends onto the Kleisli category of the monad $\mathbb{T}$?*

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