Three-space-problem for some classes of linear topological spaces

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Abstract. We examine the so-called three-space-stability for some classes of linear topological and locally convex spaces for which this problem has not been investigated.

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0. Introduction

In the theory of linear topological spaces (*lts*), particularly locally convex spaces (*lcs*) there were a lot of investigations of various hereditary questions (if a space possesses certain property P, does its subspace, quotient-space, arbitrary product or direct sum of such spaces, inductive or projective limit and so on, possess this property). The so-called *three-space-problem* is in a certain sense an inverse question: if a short exact sequence

$$0 \to F \to E \to E/F \to 0 \tag{$*$}$$

of *lts* is given (algebraic and topological exactness is assumed) in which spaces F and E/F possess certain property P, must the space E possess the same property? If the answer to such a question is positive, then the property P is called *three-space-stable*. It is of some interest to mention that in the category of linear topological spaces F and E/F can be locally convex, without E being locally convex, i.e. the property of being locally convex is not three-space-stable in the category of *lts*. Detailed account on the three-space-stability of many properties can be found in [9] and [19].

In this paper we investigate the three-space-stability for some classes of *lts* and *lcs* which, to our knowledge, have not been considered in that sense.

1. Three-space-problem for locally topological spaces

Let us recall some definitions.

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A locally convex space E is said to be a *b-space* (*b-barrelled*) if every absolutely convex subset (resp. barrel) in it is a neighbourhood of origin whenever its intersection with each bounded absolutely convex subset B of E is a neighbourhood of origin in B; E is a D_b*-space* if it is b-barrelled and has a fundamental sequence of bounded subsets (equivalently, if it is a b-space with a fundamental sequence of bounded subsets) [18].

The respective notions in the category of linear topological spaces are called *locally topological*, *ultra-b-barrelled*, resp. σ*-locally topological* spaces [1], [14] — in the previous definitions one has just to replace an absolutely convex set by a string and a barrel by an ultrabarrel.

These properties of *lts* (*lcs*) are not three-space-stable. As an example, let us show it for D_b (resp. σ-locally topological) spaces. We shall use the Example 3.5 from the paper [19]. In it, $(X, \mathcal{Z})$ is an *lcs* and L is its subspace, such that: (a) $(L, \mathcal{Z}|L)$ is a barrelled and bornological (DF)-space; it is certainly a D_b and σ-locally topological space; (b) $(X/L, \mathcal{Z}/L)$ is a normed space; (c) $(X, \mathcal{Z})$ is not a D_b (neither a σ-locally topological) space. Indeed, the opposite assumption would lead to the contradiction in the same way as in [19], using the fact that the quotient mapping $q\colon X \to X/L$ in this case lifts bounded sets with closure — see Theorem 3.2.1 [18], resp. 4.(3) [1].

Let us show now that, similarly to (ultra) bornological and (ultra) (DF)-spaces, the three-space-problem for the mentioned classes has the positive answer under some additional hypothesis.

Proposition 1.1. (a) *Let* $(*)$ *be a short exact sequence of lts, such that* F *is locally bounded and* E/F *is locally topological. Then* E *is a locally topological space.*

(b) *If the previous sequence is in the category of lcs, such that* F *is seminormed and* E/F *is a b-space, then* E *is a b-space, too.*

Proof: We shall prove the part (a); for (b) it can be done in an even simpler way. We shall use ideas as in [19].

Let $(T_n)_1^\infty$ be a locally topological string in E, i.e. let for each $B \in \mathcal{B}(E)$ (the set of all absolutely convex bounded subsets of E) and for each n, $T_n \cap B$ be a neighbourhood of origin in B in the induced topology. Then $(T_n \cap F)_1^\infty$ is a locally topological string in F. As far as the space F, being locally bounded, is locally topological, $(T_n \cap F)_1^\infty$ is a topological string, i.e. there exists a topological string $(U_n)_1^\infty$ in E, such that $F \cap (U_n + U_n) \subset T_n$ for each n. It can be assumed that the sets $F \cap (U_n + U_n)$ are bounded. Consider the string $(q(T_n \cap U_n))_1^\infty$ in E/F (q — the quotient map) and let us show that it is locally topological. Let C be an arbitrary bounded subset of E/F. According to Proposition 3.1 (a) [19], there is $B \in \mathcal{B}(E)$, such that $C = q(B)$. It follows that

$$C \cap q(T_n \cap U_n) = q(B) \cap q(T_n \cap U_n) \supset q(B \cap T_n \cap U_n),$$

hence, $C \cap q(T_n \cap U_n)$ is a neighbourhood of origin in C, since q is an open mapping. Therefore, $(q(T_n \cap U_n))_1^\infty$ is a topological string in E/F, and so, e.g.,

$q(T_2 \cap U_2)$ is a neighbourhood of origin in E/F. Now $V = U_2 \cap ((T_2 \cap U_2) + F)$ is a neighbourhood of origin in E. Since

$$V \subset U_2 \cap (T_2 \cap U_2 + F \cap (U_2 + U_2)) \subset T_2 + T_2 \subset T_1,$$

T_1 is a neighbourhood of origin in E, too, which proves that the space E is locally topological. □

In the similar way one can prove the following

Proposition 1.2. *Let* $(*)$ *be a short exact sequence of lcs (lts) in which F is seminormed (locally bounded) and E/F is (ultra) b-barrelled. Then E is (ultra) b-barrelled, too.*

Proposition 1.3. *Let* $(*)$ *be a short exact sequence of lcs (lts) in which F is seminormed (locally bounded) and E/F is a D_b (resp. σ-locally topological) space. Then E is a D_b (resp. σ-locally topological) space, too.*

Proof: Follows by combining our Proposition 1.1 and Proposition 3.3 (a) of [19]. □

2. Three-space-problem for (HM)-spaces

From [19] we know that some properties are three-space-stable in the class of Fréchet spaces, not being stable in general. It was shown in [5] that *distinguishedness* and *density condition* (DC) of S. Heinrich are three-space-stable in the class of (F)-spaces, with additional condition of lifting of bounded sets (with closure).

Recall that an *lts* E is called an *(HM)-space* if it has all isomorphic ultrapowers in the sense of S. Heinrich [11] or (equivalently) if it has all isomorphic nonstandard hulls in the sense of Henson-Moore [12], [13]. In the general case, a space E is (HM) if and only if $\mathcal{P}(E) = \mathcal{B}(E)$ ($\mathcal{P}(E)$ is the class of all precompact subsets of E) and E satisfies (DC) [11], [15]. If E is a complete *lcs*, then it is an (HM)-space if and only if it is inductively semireflexive [3], [7] and if and only if each almost Cauchy ultrafilter converges in E as soon as it converges weakly.

For (HM)-spaces we have

Proposition 2.1. *Let* $(*)$ *be a short exact sequence in the class of (F)-spaces. Then E is an (HM)-space if F and E/F are such.*

Proof: A Fréchet space is by [12] an (HM)-space if and only if it is Montel. The proposition follows from Proposition 4.4 [19]. □

Notice that a part of the converse is not true by the famous Köthe-Grothendieck Fréchet-Montel example [16; 31.5] — a quotient of a Fréchet-Montel space need not be Fréchet-Montel.

A bit stronger conclusion is valid for D_b (resp. σ-locally topological) spaces (a fortiori for (ultra) (DF)-spaces). Namely, according to Proposition 4.4 [2], Proposition 5.5 [15], and to 4.(7) and 4.(8) [1], in these classes a space is of the type (HM) if and only if it is (ultra) Schwartz. Using Proposition 3.7 [19] and the fact that subspaces and quotients of Schwartz spaces are of the same kind, we obtain

Proposition 2.2. *Let* $(*)$ *be a short exact sequence in the class of* D_b (*resp.* σ*-locally topological*) *spaces. Then* E *is an* (*HM*)*-space if and only if* F *and* E/F *are such.*

Remark 1. Particularly, in the mentioned classes every quotient of an (HM)-space is again an (HM)-space, which is not true in the class of Fréchet spaces by the mentioned example from [16]. Of course, each subspace of an (HM)-space is an (HM)-space [15] and Schwartz space is of the type (HM) [13].

From [2], [13], [15] we know that a Fréchet, resp. (DF) or ultra-(DF) space E is an (HM)-space if and only if $\mathcal{P}(E) = \mathcal{B}(E)$. According to [10], an *lcs* E is Schwartz if and only if $\mathcal{P}(E) = \mathcal{B}(E)$ and E is quasinormable. Recall that E is a *(df)-space* (*sequentially-(DF)* in some papers) if it has a fundamental sequence of bounded sets and in its strong dual E'_β each 0-sequence is equicontinuous. For this class let us prove the following

Lemma 2.3. *Let* E *be a* (*df*)*-space and consider the following properties:* 1° E *is an* (*HM*)*-space;* 2° $\mathcal{P}(E) = \mathcal{B}(E)$*;* 3° E'_β *is of the type* (*HM*).

Then $1^\circ \iff 2^\circ$ *and* $3^\circ \implies 1^\circ$.

Proof: If $\mathcal{P}(E) = \mathcal{B}(E)$, then $E'_p = E'_\beta$ is a Fréchet space (E'_p is the dual E' equipped with the topology of precompact convergence) and each E'_p-precompact subset is equicontinuous. Then E is topologically embedded in $(E'_p)'_p$ and since $(E'_p)'_p$ is by [13] an (HM)-space, E is of the same kind, as its subspace, by [15]. Hence, $2^\circ \implies 1^\circ$. The converse is true for each *lts*.

To prove that $3^\circ \implies 1^\circ$, notice that $\mathcal{P}(E'_\beta) = \mathcal{B}(E'_\beta)$, i.e. strongly bounded subsets in the dual E' of E are equicontinuous (since precompact subsets are such for (df)-spaces). So, E is quasi-barrelled, hence a (DF)-space. By [2], E is then an (HM)-space. □

This Lemma is a slight generalization of the respective proposition for (DF)-spaces from [2]. Applying the Lemma and the fact that the property "being an (HM)-space" is three-space-stable in the class of quasinormable spaces, we obtain the following

Proposition 2.4. *If* $(*)$ *is a short exact sequence of* (*df*)*-spaces, then* E *is an* (*HM*)*-space if and only if* F *and* E/F *are of that type.*

Proof: It just remains to prove that E/F is an (HM)-space if E is of that type. Indeed, since the quotient-map $q\colon E \to E/F$ lifts bounded sets with closure, one has $\mathcal{P}(E/F) = \mathcal{B}(E/F)$. □

Remark 2. Let us prove that in the class of (df)-spaces, from 1°, i.e. 2°, it doesn't follow 3°. From [6] it follows that there exists a so-called (dF)-space which is not a (DF)-space. Namely, (dF)-space is by Corollary 1.7 [6] a complete k-space with a fundamental sequence of compact subsets, i.e. an (HM)-space which is equal to its full ultrapower ($E = (E)_{\mathcal{D}}$) [11], [15]. It is obviously a D_b space [18], a fortiori a (df)-space. If its strong dual E'_β were an (HM)-space, by the proof of $3^\circ \implies 1^\circ$, it would follow that E is a (DF)-space, which is a contradiction.

In the following diagram we have the relationship between the four classes of spaces:

$$\begin{array}{ccccc} \text{(DF)} & \Longrightarrow & D_b & \Longrightarrow & \text{(df)} \\ & & \Uparrow & & \\ & & \text{(dF)} & & \end{array}$$

while the classes of (DF) and (dF)-spaces are incomparable. So, in the class of complete D_b (resp. (df)) spaces, the property "being an (HM)-space" is equivalent to "being a (dF)-space". Concerning (dF)-spaces we have

Proposition 2.5. *If* $(*)$ *is a short exact sequence of* (*df*)*-spaces, then* E *is a* (*dF*)*-space if and only if* F *and* E/F *are of that type.*

Proof: If E is a (dF)-space, then by Proposition 1.9 [6] F and E/F are of the same kind (we assume that F is a closed subspace). Conversely, if F and E/F are (dF)-spaces, then they are complete (HM)-spaces, hence E is of the same type by the Proposition 2.4 and Proposition 1.3 [19]. □

We do not know whether the property (HM) is three-space-stable in the general case.

Remark 3. As we have already said, three-space-stability of (DC)-spaces, which are in close relationship with (HM)-spaces, was investigated in [5]. In this context the following question was posed in [4]: *is there an* (*F*)*-space* E *without* (*DC*)*, such that its strong bidual* E'' *possesses* (*DC*)? A. Peris in his dissertation showed that the answer was negative. We give here an alternate proof of this fact.

Let E be an (F)-space, such that E'' has (DC). Then E'' is a distinguished (F)-space, such that bounded subsets of its strong dual E'''_β are metrizable. According to Theorem 6 [9], the spaces E and E''/E are distinguished and the quotient-map $q\colon E'' \to E''/E$ lifts bounded sets with closure (and also without closure — Theorem 8 [9]). So we have a short exact sequence of (F)-spaces

$$0 \to E \xrightarrow{i} E'' \xrightarrow{q} E''/E \to 0$$

in which q lifts bounded sets. Applying 26.12 [17], the dual sequence

$$0 \to (E''/E)'_\beta \xrightarrow{q^t} (E'')'_\beta \xrightarrow{i^t} E'_\beta \to 0$$

is topologically exact. Since E'''_β is a (DF)-space, quotient-map i^t lifts bounded sets with closure. As far as bounded sets in E'''_β are metrizable, bounded sets in E'_β have the same property, and so the space E satisfies (DC).

3. Three-space-problem for inductively semireflexive spaces

Recall the definition of *inductively semireflexive* [3], i.e. *b-reflexive spaces* [7].

Let (E, t) be an *lcs*. Denote by TE' the topology on the dual E' which has a base of the 0-neighbourhoods formed of t-equivorous discs (i.e. absolutely convex sets which absorb all t-equicontinuous subsets of E'). This topology is exactly the inductive limit topology of the family $\{ E'_{U^\circ} \mid U \in \mathcal{U}_t(0) \}$ of Banach spaces and it is finer then the strong topology $\beta(E', E)$. If $(E', TE')' = E$, then the space (E, t) is said to be inductively semireflexive. This property is in close relationship with (HM)-spaces [2], [13], and on the other hand with ultrabornological spaces [7].

Observe that the following is valid:

Lemma 3.1. *If (E, t) is a Fréchet or a (df)-space (in particular, a (DF) or a D_b space), then it is inductively semireflexive if and only if it is semireflexive.*

Proposition 3.2. *Let $(*)$ be a short exact sequence of Fréchet spaces. If F and E/F are (inductively) semireflexive, then E is of the same kind. The converse is true if the quotient-map $q\colon E \to E/F$ lifts bounded sets.*

Proof: The first part of the proposition follows from Lemma 3.1 and Proposition 4.3 [19]. As a (closed) subspace of a semireflexive space E is always of the same kind, it remains to prove that under given circumstances the quotient E/F possesses the same property. Indeed, according to 26.12 [17], the sequence

$$0 \to (E/F)'_\beta \to E'_\beta \to F'_\beta \to 0$$

is topologically exact. Then it is $F'_\beta = F'_\tau$ (since $E'_\beta/F^\circ = F'_\beta$ and $\tau(E', E)/F^\circ = \tau(F', F)$), and also $\beta(E', E)|F^\circ = \beta(F^\circ, E/F)$ (because of the exactness) and $\tau(E', E)|F^\circ \le \tau(F^\circ, E/F)$. Therefore, $\beta(F^\circ, E/F) \le \tau(F^\circ, E/F)$ and the space E/F is (inductively) semireflexive. □

Notice that in such a way in the following diagram

$$\begin{array}{ccc} \text{(HM)} & \Longrightarrow & \text{(inductively) semireflexive} \\ \Downarrow & & \Downarrow \\ \text{(DC)} & \Longrightarrow & \text{distinguished} \end{array}$$

the upper two properties are three-space-stable in the class of (F)-spaces, while the lower two are not [5].

Proposition 3.3. *If $0 \to F \xrightarrow{i} E \xrightarrow{q} E/F \to 0$ is a short exact sequence of (df)-spaces (particularly (DF) or D_b spaces), then E is (inductively) semireflexive if and only if F and E/F are of the same kind.*

Proof: Suppose that F and E/F are inductively semireflexive and let us prove that E is such. According to Lemma 3.1 and Proposition 4.2 [19] , it is sufficient to prove that in the dual F' we have $\beta(E'/F^\circ, F) = \beta(E', E)/F^\circ$. As

$(E'/F^\circ, \beta(E'/F^\circ, F))$, as metrizable, is a bornological space, each 0-sequence $\{x'_n\}$ is equicontinuous, i.e. there exists an equicontinuous subset $A \subset E'$ such that $\{x'_n\} \subset i^t(A)$, wherefrom it follows that the set $\{x'_n\}$ is $\beta(E', E)/F^\circ$-bounded.

Conversely, suppose that E is (inductively) semireflexive. Since the quotient-map q lifts bounded sets with closure, as with (DF)-spaces, we obtain that the sequence

$$0 \to (E/F)'_\beta \xrightarrow{q^t} E'_\beta \xrightarrow{i^t} F'_\beta \to 0$$

is topologically exact. As in the proof of Proposition 3.2 we conclude that the space E/F is (inductively) semireflexive. □

Remark 1. The previous proposition is a slight generalization of Proposition 4.3 [19].

Remark 2. Semireflexivity is a property which depends only on the dual pair, while inductive semireflexivity depends on the topology of the given space. So, for the spaces mentioned in Lemma 3.1 semireflexivity and inductive semireflexivity are equivalent only for the given topology. For example, if (E, t) is a semireflexive space, then (E, p) has the same property for each topology p which is compatible with the dual pair $\langle E, E' \rangle$. For inductive semireflexivity it is not the case.

4. Three-space-problem for spaces with minimal or the finest topology

As is well known, it is said that an *lts* E has *minimal topology* [20] if there is no strictly coarser linear (Hausdorff) topology on E. The minimal topology on E will be denoted by t_{mE}.

Proposition 4.1. *Let* $(*)$ *be a short exact sequence of lts. Then* E *has minimal topology if* F *and* E/F *have minimal topologies.*

Proof: Let F and E/F have minimal topologies t_{mF}, resp. $t_{m(E/F)}$ and let t be the topology of E. According to [19], t is a Hausdorff topology and $t_{mE} \le t$. Furthermore, $t_{mE}|F \le t_{mF} = t_F$ and $t_{mE}/F \le t/F = t_{m(E/F)}$; also $t_{mF} \le t_{mE}|F$ (because of minimality) and $t_{m(E/F)} \le t_{mE}/F$, so we have the equality of topologies $t = t_{mE}$ on the base of [19, p. 23]. □

Remark. The converse of the previous proposition is true in the category of locally convex spaces (see Example 6, Chapter IV of [20]). According to [8] it is an open question whether a quotient of an *lts* with minimal topology is again with minimal topology.

Proposition 4.2. *Let* $(*)$ *be a short exact sequence of lts* (*lcs*). *Then the space* E *has the finest linear* (*locally convex*) *topology if and only if* F *and* E/F *obeys this property.*

Proof: The finest locally convex topology on the space X will be denoted by T_X. Suppose that spaces F and E/F have topologies T_F and $T_{E/F}$, respectively. If t is the topology of the space E, then from $t \leq T_E$ it follows $T_F = t|F \leq T_E|F = T_F$ (since the finest locally convex topology induces such a topology on a subspace) and also $t/F \leq T_E/F \leq T_{E/F} = t/F$. As in the previous proof, from $t \leq T_E$, $t|F = T_E|F$ and $t/F = T_E/F$ it follows that $t = T_E$.

The converse is Example 7 in Chapter II of [20].

The proof is the same for the finest linear topology. $\square$

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