Combinatorics and quantifiers

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Abstract. Let $\binom{I}{m}$ be the set of subsets of I of cardinality m. Let f be a coloring of $\binom{I}{m}$ and g a coloring of $\binom{I}{m}$. We write $f \to g$ if every f-homogeneous $H \subseteq I$ is also g-homogeneous. The least m such that $f \to g$ for some $f : \binom{I}{m} \to k$ is called the k-*width* of g and denoted by $w_k(g)$. In the first part of the paper we prove the existence of colorings with high k-width. In particular, we show that for each $k > 0$ and $m > 0$ there is a coloring g with $w_k(g) = m$. In the second part of the paper we give applications of wide colorings in the theory of generalized quantifiers. In particular, we show that for every monadic similarity type $t = (1, \dots , 1)$ there is a generalized quantifier of type t which is not definable in terms of a finite number of generalized quantifiers of a smaller type.

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1. The width of a coloring

Let $\binom{I}{m}$ be the set of all subsets of I of cardinality m. (Thus $\binom{I}{m} = \emptyset$ for $|I| < m$.) The set I is thought to be either infinite or a large finite set. A mapping $f : \binom{I}{m} \to k$, where k is finite, is called a *coloring*. A set $H \subseteq I$ is called f*-homogeneous* if f restricted to the set $\binom{H}{m}$ is a constant mapping.

Let f be a coloring of $\binom{I}{m}$ and g a coloring of $\binom{I}{n}$. The following is the principal relation investigated in this paper: We write $f \to g$ if every f-homogeneous $H \subseteq I$ is g-homogeneous.

One can easily see that the relation "$\to$" is a quasiorder. Observe also that for $m > n$ the relation $f \to g$ implies that g is a constant mapping. Thus for $m \neq n$ the relation $f \to g \to f$ is equivalent to both f and g being constant (i.e. $|I|$ being both f- and g-homogeneous). Because of this we assume $m \leq n$ when considering the relation $f \to g$.

Here is another less trivial example: If

$$g(\{\alpha_1, \dots , \alpha_n\}) = f(\{\alpha_1, \dots , \alpha_m\}),$$

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whenever $\{\alpha_1, \ldots, \alpha_n\} \in \binom{I}{n}$ and $\alpha_1 < \ldots < \alpha_n$, then $f \to g$.

The least m such that $f \to g$ for some $f : \binom{I}{m} \to k$ is called the *k-width* of g and denoted by $w_k(g)$. If I is infinite, then the *width* $w(g)$ of g is the number $\min_{k<\omega} w_k(g)$.

The main question we study in this chapter is: *How to construct wide colorings?*

First we consider the first non-trivial case of the width, i.e. 1. We can think of a 2-coloring $g : \binom{I}{n} \to \{0, 1\}$ as a hypergraph $G = (I, E)$, where E is the set of sets with color 1. In this case we denote G by $\hat{g}$. The *cochromatic* number $z(G)$ of an n-uniform hypergraph is the least k so that for some k-coloring of G every color class is either edgefree or complete ([4]). (A set K is complete in (I, E) if $\binom{K}{n} \subseteq E$.) We use $\chi(G)$ to denote the chromatic number of G.

Theorem 1. *The following conditions are equivalent for any 2-coloring g of $\binom{I}{n}$, $|I|$ infinite, and any k:*

(1) $w_k(g) \leq 1$.
(2) $z(\hat{g}) \leq k$.
(3) *There are complete subgraphs $H_1, \ldots, H_l$, $l \leq k$, so that if they are removed from $\hat{g}$, leaving H, then $\chi(H) \leq k - l$.*

Proof: To prove that (1) implies (2), suppose an f witnessing (1) exists. Then f colors $\hat{g}$ with k colors. If some color class is neither edgefree nor complete, then $f \not\to g$. Hence $z(\hat{g}) \leq k$. It is obvious that (2) implies (1). To prove that (2) implies (3), suppose f is a k'-coloring of $\hat{g}$, witnessing $z(\hat{g}) = k' \leq k$. Remove the l ($\leq k'$) complete color classes from $\hat{g}$, obtaining H. The remaining ones are edgefree. Hence $\chi(H) \leq k' - l \leq k - l$. Finally, to prove that (3) implies (1), suppose H is as in (3). Suppose f is a $k-l$-coloring of H witnessing $\chi(H) \leq k-l$. Extend f to the l removed cliques getting a k-coloring f' of $\hat{g}$. Thus $z(\hat{g}) \leq k$. □

Corollary 2. *The following conditions are equivalent for any 2-coloring g of $\binom{I}{n}$, $|I|$ infinite:*

(1) $w(g) \leq 1$.
(2) $z(\hat{g}) < \omega$.
(3) *There are complete subgraphs $H_1, \ldots, H_l$, $l < \omega$, so that if they are removed from $\hat{g}$, leaving H, then $\chi(H) < \omega$.*

Using the above characterization we get numerous explicit examples of 2-colorings g of $\binom{\omega}{n}$ of width 2.

Example 3. Let $n \geq 2$ and let $\{A_i : i \in \omega\}$ be a partition of ω into infinitely many infinite classes. Let

$$g(\{x_1, \ldots, x_n\}) = \begin{cases} 1 & \text{if } \{x_1, \ldots, x_n\} \subseteq A_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{g}$ is an infinite union of infinite cliques, hence $z(\hat{g}) = \omega$, and therefore $w(g) \geq 2$, for every $n \geq 2$. Actually, it is easy to see that $w(g) = 2$. For example, if we let

$$g(\{x, y\}) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ have the same least prime factor,} \\ 0 & \text{otherwise,} \end{cases}$$

then $w(g) = 2$.

Example 4. Let $n \geq 2$ and let $\{A_i : i \in \omega\}$ be a partition of ω so that $\lim_{n\to\infty}(|A_i| - i) = \infty$. Let

$$g(\{x_1, \dots, x_n\}) = \begin{cases} 1 & \text{if } \{x_1, \dots, x_n\} \subseteq A_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $z(\hat{g}) = \omega$, and therefore $w(g) \geq 2$ for every $n \geq 2$. Again, it is easy to see that $w(g) = 2$. For example, we could choose

$$g(\{x, y\}) = \begin{cases} 1 & \text{if } [\sqrt{x}] = [\sqrt{y}]. \\ 0 & \text{otherwise,} \end{cases}$$

and then $w(g) = 2$.

Example 5. Suppose G is the union of $G_i = (G_i, E_i)$, $i < \omega$, so that $\lim_{n\to\infty} z(G_i) = \infty$. Let

$$g(\{x_1, \dots, x_n\}) = \begin{cases} 1 & \text{if } \{x_1, \dots, x_n\} \in E_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $z(\hat{g}) = \omega$, and therefore $w(g) \geq 2$. For example, we could have

$$g(\{x, y\}) = \begin{cases} 1 & \text{if } x \leq y \leq 2^x, \\ 0 & \text{otherwise,} \end{cases}$$

and then $w(g) = 2$.

Now we discuss width 3. Although more complicated we still have a large variety of width 3 colorings.

Theorem 6. *Suppose*

$$g(\{x, y, z\}) = \begin{cases} 1, & \textit{if } x < y < z < \omega \textit{ and } y - x < z - y, \\ 0, & \textit{otherwise.} \end{cases}$$

Then $w(g) = 3$.

Proof: Let us suppose first the following Ramsey type result:

> (∗) For any c there is an n so that for any coloring of $[n]^2$ with c colors there is a homogeneous set $\{x_1 < \dots < x_4\}$ with
>
> $$x_2 - x_1 < x_4 - x_3 < x_3 - x_2.$$

Then the theorem follows: Suppose a coloring f of $\binom{\omega}{2}$ is given with $f \to g$. Choose a large n and an f-homogeneous set $\{x_1 < \ldots < x_4\}$ with

$$x_2 - x_1 < x_4 - x_3 < x_3 - x_2.$$

Then the set $\{x_1, \ldots, x_4\}$ is not g-homogeneous. □

The second author was not able to prove $(*)$ and discussed the matter with Joel Spencer. Very soon Noga Alon [1] proved a stronger result. We present it here, with the kind permission of Noga Alon. For more on this theorem, see [11], where a doubly exponential upper bound is achieved.

Suppose $x_1 < \ldots < x_n$ are natural numbers. Let $y_i = x_i - x_{i-1}$ for $i = 2, \ldots, n$. For a permutation σ of $[2, n]$ we say that $x_1 < \ldots < x_n$ has *type* σ provided that $y_{\sigma 1} < \ldots < y_{\sigma n}$.

Theorem 7 ([1])**.** *For any $k_1, \ldots, k_r$ and any permutations $\sigma_1, \ldots, \sigma_r$ of $[2, k_1], \ldots, [2, k_r]$ there is n so that in any r-coloring of $\binom{n}{2}$ for some i, there is a homogeneous set of color i, of size k_i and of type σ_i.*

Proof: We use induction on the sum $k_1 + \ldots + k_r$. Let l be large enough so that the claim holds for K_l and any sequence with smaller sum. Assume n is large. Let $N_1, \ldots, N_l$ be disjoint intervals of integers $< n$ so that each has length about $\frac{n}{6l}$ and they are at least $\frac{n}{3l}$ apart from each other. Let f be a fixed r-coloring of $\binom{n}{2}$. Define a coloring c of $B = N_1 \times \ldots \times N_l$ by letting the color of $(x_1, \ldots, x_l)$ code the colors of all pairs $\{x_i, x_j\}$. Thus c uses $r^{\binom{l}{2}}$ colors. If we chose n large enough, then Gallai-Witt's Theorem implies that there are arithmetic progressions $A_i \subseteq N_i$, $|A_i| = l$, so that every $(x_1, \ldots, x_l) \in A_1 \times \ldots \times A_l$ has the same color. We now have an induced coloring χ of $\binom{l}{2}$: If $i \neq j$ are in $[l]$, we let the color of $\{i, j\}$ be the color of any edge between A_i and A_j. We shall apply the induction hypothesis to the coloring χ. For this purpose we reduce each permutation σ_i to a permutation σ'_i of $[3, k_i]$ by leaving out number 2 from $\mathrm{dom}(\sigma_i)$. The induction hypothesis gives a color j, call it red, and a monochromatic sequence $i_2, \ldots, i_{k_j}$ so that any edge between A_{i_u} and A_{i_v} is red. Let $\sigma_j(2) = a$.

Case 1: There are elements $b < c$ in A_a so that the edge between them is red. Let $b_u \in A_u$ for $u \neq a$. Then the homogeneous set $\{b_u : u = 1, \ldots, k_j, u \neq a\} \cup \{b, c\}$ has type σ_j.

Case 2: There are no elements $b < c$ in A_a so that the edge between them is red. In this case we have reduced the number of colors by one, and we can use induction hypothesis to the arithmetic progression A_a of length l. □

The following result gives an alternative construction of a coloring g of width 3 in the spirit of the proof in [9]:

Theorem 8. *For each k there are N and $g : \binom{N}{3} \to 2$ so that $w_k(g) = 3$.*

PROOF: Choose n so that $2^{n-1} > k$. Let N be large. We consider the cartesian product N^n. For $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n) \in N^n$, let $s(\vec{x}, \vec{y}) = (t_1, \ldots, t_n)$, where $t_i = +$ if $x_i < y_i$ and $t_i = -$, if $x_i \geq y_i$. Let $t(\vec{x}, \vec{y}) = \{s(\vec{x}, \vec{y}), s(\vec{y}, \vec{x})\}$. Define $g : \binom{N^n}{3} \to 2$ by

$$g(\{\vec{x}, \vec{y}, \vec{z}\}) = \begin{cases} 1 & \text{if } t(\vec{x}, \vec{y}) = t(\vec{y}, \vec{z}) = t(\vec{x}, \vec{z}), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $f : \binom{N^n}{2} \to k$ is arbitrary. By Ramsey's Theorem there are $C_1, \ldots, C_n$ so that $|C_i| \geq 4$ and $f(\{\vec{x}, \vec{y}\})$ depends on $t(\vec{x}, \vec{y})$ only for distinct $\vec{x}, \vec{y} \in C_1 \times \ldots \times C_n$. Say

$$f(\{\vec{x}, \vec{y}\}) = \pi(t(\vec{x}, \vec{y})).$$

Since there are 2^{n-1} sets of the form $t(\vec{x}, \vec{y})$ and only k colors, there are two sets $T_1 \neq T_2$ with $\pi(T_1) = \pi(T_2)$. It is easy to construct $\vec{x_1}, \ldots, \vec{x_4} \in C_1 \times \ldots \times C_n$ so that $t(\vec{x_1}, \vec{x_2}) = t(\vec{x_2}, \vec{x_3}) = t(\vec{x_1}, \vec{x_3}) = T_1$, but $t(\vec{x_1}, \vec{x_4}) = T_2$. Hence $\{\vec{x_1}, \ldots, \vec{x_4}\}$ is f-homogeneous but not g-homogeneous. $\square$

When we look for colorings of width > 3, there is a "very simple" argument on uncountable domains: Let $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{exp_n(\kappa)}$.

Theorem 9. *Let $\kappa = (\exp_{n-1}(\omega))^+$. For every n there is a coloring $g : \binom{\kappa}{n+1} \to 2$ so that $w(g) = n + 1$.*

PROOF: We may assume $n > 0$. It is known that

$$\kappa \not\to (\aleph_1)^{n+1}_2.$$

Let g be a coloring of $[\kappa]^{n+1}$ with two colors but without an uncountable homogeneous set. Suppose $f \to g$ for some $f : \binom{\kappa}{n} \to \omega$. By the Erdös-Rado theorem

$$\kappa \to (\aleph_1)^n_\omega$$

we can find an uncountable $H \subseteq \kappa$, which is f-homogeneous. This set H cannot, however, be g-homogeneous, as g has no uncountable homogeneous sets what-so-ever.

$\square$

For finite domains the problem is not so simple and we have to invoke the Structural Ramsey Theorem, see [10] [8]. The Structural Ramsey Theorem implies the validity of Ramsey theorem for partitions of substructures (such as n-sets) and guarantees a homogeneous (induced) substructure (such as $\binom{n+2}{n}$ with an extra $(n+1)$-tuple) while avoiding a given irreducible structure (such as $\binom{n+2}{n+1}$). In the following proof we use a very special form of this result:

Theorem 10. *For each n and k there is $g : \binom{\omega}{n+1} \to 2$ so that $w_k(g) = n+1$.*

PROOF: By [10] there are $M \subseteq \binom{\omega}{n}$ and $M' \subseteq \binom{\omega}{n+1}$ so that:

1. For each k and for each $f : M \to k$ there exists an f-homogeneous $Y \subseteq \omega$ with $|Y| = n+2$ and $\binom{Y}{n+1} \cap M' \neq \emptyset$.
2. If $Y \subseteq \omega$ with $|Y| = n+2$, then $\binom{Y}{n+1} \not\subseteq M'$.

We define $g : \binom{\omega}{n+1} \to k$ by $g(\{x_1, \dots, x_{n+1}\}) = 1$, if $\{x_1, \dots, x_{n+1}\} \in M'$, and $g(\{x_1, \dots, x_{n+1}\}) = 0$ otherwise. To prove that g is the coloring we need, suppose $f : \binom{\omega}{n} \to k$ is arbitrary. Let Y be as in condition 1 above. By condition 2, Y is not g-homogeneous. □

2. Definability of generalized quantifiers

A *unary structure* $\mathbf{A} = (A, P_1, \dots, P_n)$ consists of a set A together with some subsets $P_1, \dots, P_n$ of A. We call the number n the *width* of $\mathbf{A}$. We denote the class of all unary structures of width n by $\mathrm{Str}(n)$. The unary structure $\mathbf{A}$ is called *basic* if the subsets $P_1, \dots, P_n$ are disjoint. We can associate with a unary structure $\mathbf{A}$ of width n a basic structure of width $2^n - 1$ by considering intersections of the sets P_i and their complements. The old subsets and the new subsets are definable from each other in an obvious way.

A *unary quantifier* of width n is any collection Q of unary structures of width n so that Q is closed under isomorphisms. If Q consists of basic structures, it is called *basic*. This concept is due to Mostowski [7] for $n = 1$ and to Lindström [5] for $n > 1$.

Here are some examples of unary quantifiers:

1. $\exists = \{(A, P) : \emptyset \neq P \subseteq A\}$ and $\forall = \{(A, P) : P = A\}$ are basic unary quantifiers of width 1.
2. $Q_\alpha = \{(A, P) : P \subseteq A, |P| \geq \aleph_\alpha\}$ is a unary quantifier of width 1.
3. The *Rescher-quantifier* $J = \{(A, B, C) : B, C \subseteq A, |B| \leq |C|\}$ is a unary quantifier of width 2. The related quantifier $J' = \{(A, B, C, D) : A, B, C$ and D disjoint, $|B \cup C| \leq |C \cup D|\}$ is a basic unary quantifier of width 3. Note that
$$(A, B, C) \in J \iff (A, B \setminus C, B \cap C, C \setminus B) \in J'$$
and
$$(A, B, C, D) \in J' \iff (A, B \cup C, C \cup D) \in J.$$

The *definability* of one quantifier in terms of others is defined by introducing a formal language (following [5] and [7]). We present an outline of the definition of this language for completeness:

Definition 11. *Suppose $Q_1, \dots, Q_n$ are quantifier of widths $m_1, \dots, m_n$, respectively. The* first order language with the unary quantifiers $Q_1, \dots, Q_n$, *in symbols $\mathcal{L}_{\omega\omega}(Q_1, \dots, Q_n)$ consists of atomic formulas $x_i = x_j$, $\mathbf{P}_i(x_j)$ and the* *combined formulas obtained by conjunction $\phi \wedge \psi$, negation $\neg\phi$, existential quantification $\exists x_i\phi$ and Q_i-quantification $\mathbf{Q}_i x_1 \dots x_{n_i}\phi_1 \dots \phi_{n_i}$. The truth $\mathbf{A} \models \phi(\mathbf{a})$, $\mathbf{a} = (a_1, \dots, a_m)$, of a formula $\phi(x, \dots, x_m)$ in a structure $\mathbf{A} = (A, P_1, P_2, \dots)$ under the interpretation $x_i \mapsto a_i$ of variables is defined with the conditions:*

$$
\begin{aligned}
&\mathbf{A} \models (x_i = x_j)(\mathbf{a}) \iff a_i = a_j,\\
&\mathbf{A} \models \mathbf{P}_i(x_j)(\mathbf{a}) \iff a_i \in P_i,\\
&\mathbf{A} \models (\phi \wedge \psi)(\mathbf{a}) \iff \mathbf{A} \models \phi(\mathbf{a}) \textit{ and } \mathbf{A} \models \psi(\mathbf{a}),\\
&\mathbf{A} \models (\neg\phi)(\mathbf{a}) \iff \mathbf{A} \not\models \phi(\mathbf{a}),\\
&\mathbf{A} \models \exists x\phi(x, \mathbf{a}) \iff \{a \in A : \mathbf{A} \models \phi(a, \mathbf{a})\} \neq \emptyset\\
&\mathbf{A} \models \mathbf{Q}_i x_1 \dots x_{m_i}\phi_1(x_1, \mathbf{a}) \dots \phi_{m_i}(x_i, \mathbf{a}) \iff (A, R_1, \dots, R_{m_i}) \in Q_i,\\
&\textit{where } R_j = \{a \in A : \mathbf{A} \models \phi_j(a, \mathbf{a})\}.
\end{aligned}
$$

A quantifier Q of width n is definable in terms of *quantifiers $Q_1, \dots, Q_m$ if there is a formula ϕ in $\mathcal{L}_{\omega\omega}(Q_1, \dots, Q_m)$ so that*

$$Q = \{\mathbf{A} \in \mathrm{Str}(n) : \mathbf{A} \models_{\mathbf{a}} \phi \ \textit{ for all } \ \mathbf{a}\}.$$

For example, the quantifiers J and J' are definable in terms of each other. Indeed, every quantifier of width n is definable in terms of an obvious basic quantifier of width $2^n - 1$. This means that, up to definability, the width hierarchy of basic quantifiers is finer than that of quantifiers. The quantifiers correspond to levels $1, 3, 7, 15, 31, 63, \dots, 2^n - 1, \dots$ of the hierarchy of basic quantifiers. The topic of this paper is the problem:

The Unary Width Problem: Construct for each n a basic unary quantifier of width $n + 1$ which is not definable in terms of basic unary quantifiers of width n.

Let Q be a quantifier of width n. We define a coloring f_Q of $\binom{\omega}{n}$ as follows: Suppose $x = \{m_1, \dots, m_n\} \in \binom{\omega}{n}$ with $m_1 < \dots < m_n$. Let $\mathbf{A}_x$ be a basic unary structure $(A, P_1, \dots, P_n)$, where $|P_i| = \aleph_{m_i}$ and $|A \setminus \bigcup_{i=1}^{n} P_i| = \aleph_\omega$. Let

$$f_Q(x) = \begin{cases} 1, & \text{if } \mathbf{A}_x \in Q,\\ 0, & \text{otherwise.} \end{cases}$$

Proposition 12. *Suppose Q is a basic unary quantifier. If Q is definable in terms of basic unary quantifiers of width n, then $w(f_Q) \leq n$.*

PROOF: Suppose Q is of width t and is definable by a sentence ϕ of length k of $\mathcal{L}_{\omega\omega}(Q_1, \dots, Q_m)$, where $Q_1, \dots, Q_m$ are basic unary quantifiers of width n. The quantifiers $Q_1, \dots, Q_m$ and the number k give rise to a coloring g of $\binom{\omega}{n}$ as follows. Let $m_1 < \dots < m_n < \omega$. For any function

$$\sigma : [0, n] \to \{0, 1, \dots, k + n' + 1\}$$

let $\mathbf{B}_\sigma(m_1, \ldots, m_{n'})$ be the unary structure $(B, R_1, \ldots, R_n)$, where

$$|R_i| = \begin{cases} \sigma(i), & \text{if } \sigma(i) \le k, \\ \aleph_{m_{\sigma(i)-k}}, & \text{if } k < \sigma(i) \le k + n', \\ \aleph_\omega, & \text{if } \sigma(i) = k + n' + 1, \end{cases}$$

and

$$|B \setminus \bigcup_{i=1}^{n'} R_i| = \begin{cases} \sigma(0), & \text{if } \sigma(0) \le k, \\ \aleph_{m_{\sigma(0)-k}}, & \text{if } k < \sigma(0) \le k + n', \\ \aleph_\omega, & \text{if } \sigma(0) = k + n' + 1. \end{cases}$$

We let the color $g(\{m_1, \ldots, m_n\})$ code all triples (σ, j, d), where σ is as above, $j = 1, \ldots, m$ and

$$d = \begin{cases} 1, \text{ if } \mathbf{B}_\sigma(m_1, \ldots, m_n) \in Q_j, \\ 0, \text{ otherwise.} \end{cases}$$

To prove $g \to f_Q$, suppose there is a subset H of ω so that H is g-homogeneous but not f_Q-homogeneous. In particular, there are $x = \{m_1 < \ldots < m_t\} \subseteq H$ and $y = \{m'_1 < \ldots < m'_t\} \subseteq H$ so that $f_Q(x) \ne f_Q(y)$. Thus $\mathbf{A}_x \in Q \iff \mathbf{A}_y \notin Q$, and therefore

$$\mathbf{A}_x \models \phi \iff \mathbf{A}_y \not\models \phi.$$

Let $\mathbf{A}_x = (A, P_1, \ldots, P_t)$ and $\mathbf{A}_y = (A', P'_1, \ldots, P'_t)$.

We now prove by induction on k the following

Claim: If the length of $\psi(x_1, \ldots, x_r)$ is at most k and $\mathbf{a} = (a_1, \ldots, a_r)$ and $\mathbf{b} = (b_1, \ldots, b_r)$ are such that

$$a_i = a_j \iff b_i = b_j$$

and

$$a_i \in P_j \iff b_i \in P'_j,$$

then

$$\mathbf{A}_x \models \psi(a_1, \ldots, a_r) \iff \mathbf{A}_y \models \psi(b_1, \ldots, b_r).$$

The only interesting induction step is that arising from one of the quantifiers Q_j. Suppose therefore that $\mathbf{A}_x \models \mathbf{Q}_j x_1 \ldots x_n \phi_1(x_1, \mathbf{a}) \ldots \phi_n(x_n, \mathbf{a})$. Let $R_i = \{a \in A : \mathbf{A}_x \models \phi_i(a, \mathbf{a})\}$ and $R'_i = \{b \in A : \mathbf{A}_y \models \phi_i(b, \mathbf{b})\}$. Let $m_0 = m$, $R_0 = A \setminus \bigcup_i R_i$, and $P_0 = A \setminus \bigcup_i P_i$. Note that each set R_i is closed under automorphisms of $\mathbf{A}_x$ that fix $\mathbf{a}$ pointwise. Hence there is a mapping $h : [0, t] \to [0, n]$ so that if $S_i = \bigcup\{P_j : h(j) = i\}$, then $R_i \triangle S_i \subseteq \{a_1, \ldots, a_r\}$. If $S'_i = \bigcup\{P'_j : h(j) = i\}$, then, by Induction Hypothesis, $R'_i \triangle S'_i \subseteq \{b_1, \ldots, b_r\}$. Note that

$|P_j| = \aleph_{m_j}$ and $|S_i| = 0$ or $|S_i| = \aleph_{m_i^*}$, where $m_i^* = \max\{m_j : h(j) = i\}$. Similarly, $|P'_j| = \aleph_{m'_j}$ and $|S'_i| = 0$ or $|S'_i| = \aleph_{m_i'^*}$, where $m_i'^* = \max\{m'_j : h(j) = i\}$. Let π be a permutation of $[0, n]$ with

$$S_{\pi i} = \emptyset \iff i \in \{1, \ldots, l\}, \text{ and } m^*_{\pi(l+1)} < \ldots < m^*_{\pi n} < m^*_{\pi 0} = \omega.$$

Then also

$$S'_{\pi i} = \emptyset \iff i \in \{1, \ldots, l\}, \text{ and } m'^*_{\pi(l+1)} < \ldots < m'^*_{\pi n} < m'^*_{\pi 0} = \omega.$$

We define $\sigma : [0, n] \to \{0, 1, \ldots, k + n - l + 1\}$ as follows:

$$\sigma(i) = \begin{cases} |R_i|, & \text{if } |R_i| \leq k, \\ k + \pi^{-1}(i), & \text{if } k < |R_i| < \aleph_\omega, \\ k + n - l + 1, & \text{if } |R_i| = \aleph_\omega. \end{cases}$$

Now,

$$(A, R_1, \ldots, R_n) \cong \mathbf{B}_\sigma(m^*_{\pi(l+1)}, \ldots, m^*_{\pi n}) \in Q_j.$$

Respectively,

$$(A, R'_1, \ldots, R'_n) \cong \mathbf{B}_\sigma(m'^*_{\pi(l+1)}, \ldots, m'^*_{\pi n}).$$

Since H is g-homogeneous, $\mathbf{B}_\sigma(\beta^*_{\pi(l+1)}, \ldots, \beta^*_{\pi n}) \in Q_j$. Since Q_j is closed under isomorphisms, we have $(A, R'_1, \ldots, R'_n) \in Q_j$, or equivalently, $\mathbf{A}_y \models \mathbf{Q}_j x_1 \ldots x_n \phi_1(x_1, \mathbf{b}) \ldots \phi_n(x_n, \mathbf{b})$.

By letting $\psi(x_1, \ldots, x_r)$ be the sentence ϕ in the claim, we get a contradiction, and the theorem is proved. $\square$

Lindström [5] proved that the Rescher-quantifier is not definable in terms of quantifiers of width 1. His proof was based on the observation that using the Rescher-quantifier one can define well-ordering implicitly, while this is not possible using quantifiers of width 1 only. Subsequently many unary quantifiers of width 2 have been shown to be undefinable in terms of quantifiers of width 1, even on finite structures (see [3], [12]). We can get many more by means of Theorem 1.

For example, the following basic unary quantifiers of width 2 are not definable in terms of basic unary quantifiers of width 1:

$$\begin{aligned} Q = \{(A, P_1, P_2) : \; & P_1 \cap P_2 = \emptyset, \\ & |A| = \aleph_\omega, |P_1| = \aleph_n, |P_2| = \aleph_m, \text{ and } S(n, m) \text{ holds}\}, \end{aligned}$$

where $S(n, m)$ may be any of the following predicates:

$$n \text{ and } m \text{ have the same smallest prime factor}$$

$$[\sqrt{n}] = [\sqrt{m}]$$

$$n \leq m \leq 2^n.$$

Theorem 13. *The basic quantifier* $Left = \{(A, B, C, D)$: *there are more cardinals between* $|B|$ *and* $|C|$ *than between* $|C|$ *and* $|D|\}$ *of width* 3 *is not definable in terms of basic quantifiers of width* 2.

Proof: Follows from Theorem 8. □

Suppose $n > 1$ and let g_n be the coloring of width n given by Theorem 10. We define a basic unary quantifier R_n of width n as follows:

$$\mathbf{A} \models R_n x_1 \dots x_n \phi_1(x_1) \dots \phi_n(x_n)$$

if the formulas are pairwise disjoint and for some $m_1 < \dots < m_n \in \omega$ so that $g_n(\{m_1, \dots, m_n\}) = 1$ we have $|\phi_1^{\mathbf{A}}| = \aleph_{m_1}, \dots, |\phi_n^{\mathbf{A}}| = \aleph_{m_n}$.

Theorem 14. *The basic unary quantifier* R_{n+1} *of width* $n+1$ *cannot be defined in terms of any finite number of basic unary quantifiers of width* n.

Proof: Follows from Theorem 9. □

Theorem 14 gives a full solution to the Unary Width Problem. Other solutions have been obtained, independently, by Kerkko Luosto [6] and Per Lindström [13]. The proof of Lindström uses a counting argument and it gives the result on finite models, too. This method has been further developed in [2]. Other results about the Unary Width Problem on finite structures can be found in [3] and [12].

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