Finite canonization

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Abstract. The canonization theorem says that for given m, n for some m^* (the first one is called $ER(n;m)$) we have

> for every function f with domain $[1, \dots, m^*]^n$, for some $A \in [1, \dots, m^*]^m$, the question of when the equality $f(i_1, \dots, i_n) = f(j_1, \dots, j_n)$ (where $i_1 < \cdots < i_n$ and $j_1 < \cdots j_n$ are from A) holds has the simplest answer: for some $v \subseteq \{1, \dots, n\}$ the equality holds iff $\bigwedge_{\ell \in v} i_\ell = j_\ell$.

We improve the bound on $ER(n, m)$ so that fixing n the number of exponentiation needed to calculate $ER(n, m)$ is best possible.

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§0. Introduction

On Ramsey theory see the book Graham Rothschild Spencer [GrRoSp]. This paper is self-contained.

The canonical Ramsey theorem was originally proved by Erdös and Rado, so the relevant number is called $ER(n, m)$. See [ErRa], [Ra86] and more in the work of Galvin. The theorem states that if m and n are given, and f is an n-place function on a set A of size $\geq ER(n, m)$, then there is an $A' \in [A]^m$ such that f is canonical on A'. That is, for some $v \subseteq \{1, \dots, n\}$ and for every $i_1 < \cdots < i_n \in A'$ and $j_1 < \cdots < j_n \in A$

$$f(i_1, \dots, i_n) = f(j_1, \dots, j_n) \Leftrightarrow \bigwedge_{\ell \in v} i_\ell = j_\ell.$$

Galvin got in the early seventies by the probability method a lower bound which appeared in [ErSp, p. 30], $ER(2;m) \geq (m + o(1))^m$.
Lefmann and Rödl [LeRo93] proved

$$2^{cm^2} < ER(2;m) \leq 2^{2^{c_1^{m^3}}}.$$

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Lefmann and Rödl [LeRo94] proved:

(i) $2^{c_2 m^2} \leq ER(2;m) \leq 2^{c_2^*(m^2 \log m)}$

(ii) $\beth_n(c_k m^2) \leq ER(n+1;m) \leq \beth_{n+1}\left(c_k^* \frac{m^{2k-1}}{\log m}\right)$.

See more on this in [LeRo94] and below for the definition of $\beth_n$.
We thank Nešetřil for telling us the problem; which for us was finding the right number of exponents (i.e. the subscript for $\beth$ in (ii) above) in $ER(n;m)$ (for a fixed n). We prove here that this number is n.
Why is the number of exponentiations best possible? Let $r_t^n(m)$ be the first r such that: $r \to (m)_t^n$, now trivially $ER(n;m) \geq r_t^n(m)$ when m is not too small, and $r_t^n(m)$ needs $n-1$ exponentiations when t is not too small.

§1. The finitary canonization lemma

Notation.. $\mathbb{R}$, $\mathbb{N}$ are the set of reals and natural numbers respectively. The letters k, ℓ, m, n will be used to denote natural numbers, as well as $i, j, \alpha, \beta, \gamma, \zeta, \xi$. We let ε be a real (usually positive).
If A is a set,

$$[A]^n = \{u \subseteq A : |u| = n\}.$$

We call finite subsets u, v of $\mathbb{N}$ neighbors if:

$$|u| = |v|, \ |u \backslash v| = 1$$

and

$$[k \in u \backslash v, \ell \in v \backslash u, m \in u \cap v \Rightarrow k < m \equiv \ell < m].$$

For $m \in \mathbb{N}$, we let $[m] = \{1, \ldots, m\}$.
For a set A of natural numbers and $i \in \mathbb{N}$, $A < i$ means $(\forall j \in A)(j < i)$. We similarly define $i < A$.

With i, A as above

$$A_{>i} \text{ denotes the set } \{j \in A : j > i\}.$$

We use the convention that $A_{>\sup \emptyset}$ is A.
Let $\beth_n(m)$ be defined by induction on n : $\beth_0(m) = m$ and $\beth_{n+1}(m) = 2^{\beth_n(m)}$.
Usually, c_i denotes a constant. □

1.1 Lemma (Finitary Canonization). *Assume n is given, then there is a constant c computable from n, such that if m is large enough:*
If f is an n-place function from $[m^\otimes] = \{1, \ldots, m^\otimes\}$ and $m^\otimes > \beth_{n-1}(cm^{8(2n-1)})$ <u>then</u> for some $A' \in [\{1, \ldots, m^\otimes\}]^m$, f is canonical on A'; i.e. for some $v \subseteq \{1, \ldots, n\}$ for every $i_1 < \cdots < i_n$ from A' and $j_1 < \cdots < j_n$ from A', we have

$$f(i_1, \ldots, i_n) = f(j_1, \ldots, j_n) \Leftrightarrow \bigwedge_{\ell \in v} i_\ell = j_\ell.$$

The proof is broken into several claims.

Explanation of our proof.

We inductively on $n^* = n^{\otimes}, \dots, 1$ decrease the set to A_{n^*} while increasing the amount of "partial homogeneity", i.e. conditions close to: results of computing f on an n-tuple from A_{n^*} are not dependent on the last $p = n^{\otimes} - n^*$ members of the n-tuple. Having gone down from $n^{\otimes}$ to n^*, we want that: if $u_1, u_2 \in [A_{n^*}]^n$ are neighbors differing in the ℓ-th place element only then: if $\ell < n^*$, the truth value of $f(u_1) = f(u_2)$ depends on the first n^* elements of u_1 and u_2 only; if $\ell > n^*$ the truth value of $f(u_1) = f(u_2)$ depends on the first n^* elements of u_1 only. Lastly if $\ell = n^*$, it is little more complicated to control this; but the truth value is monotonic and we introduce certain functions, (the h's) which express this. Arriving to $n^* = 1$ we eliminate the h's (decreasing a little) so we get the sufficiency of the condition for equality, but we still have the necessity only for u_1, u_2 which are neighbors. Then by random choice (as in [Sh37]), we get the necessity for all pairs of sets. The earlier steps cost essentially one exponentiation each, the last two cost only taking a power.

1.2 Claim. *Assume*

$(*)_0$ $m \geq 2^{(1+\varepsilon)c_1(m^*)^{n^*}}$
$t > 0$, $n^* > 1$, $k(*) > 0$ (c_1 is defined in the proof from $k(*)$, n^*)
and m^* is large enough (relative to $1/\varepsilon, t, k(*), n^*$)

$(*)_1$ $A \subseteq \mathbb{N}$, $|A| > m$
f_k a function with domain $[A]^{n^*}$ for $k < k(*)$,
h_k is a function from $[A]^{n^*}$ to $\mathbb{N}$ for $k < k(*)$, and
g is a function with domain $[A]^{n^*}$ such that Rang(g) has cardinality $\leq t$.

Then we can find A^*, j^* such that:

$(*)_2$ *$A^* \subseteq A$, $|A^*| > m^*$ and $j^* \in A_{>\sup(A^*)}$ and we have:*
if $k < k(*)$, $u \in [A^*]^{n^*-1}$ and $v \in [A^*]^{n^*-1}$, then

(α) *if u, v are neighbors, then for all $i \in A^*_{>\sup(u \cup \nu)}$ we have*

$$f_k(u \cup \{i\}) = f_k(v \cup \{i\}) \Leftrightarrow f_k(u \cup \{j^*\}) = f_k(v \cup \{j^*\})$$

(β) *if $u = v$ then for every $i_0 < i_1$ from $A^*_{>\sup(u)}$ we have*[1]

$$f_k(u \cup \{i_0\}) = f_k(u \cup \{i_1\}) \Leftrightarrow f_k(u \cup \{i_0\}) = f_k(u \cup \{j^*\})$$

(γ) *for all $i \in A^*_{>\sup(u)}$*

$$g(u \cup \{i\}) = g(u \cup \{j^*\})$$

[1]This is used later to define the h_k for the "next step".

(δ) *either for all* $i \in A^*_{>\sup(u)} \cup \{j^*\}$ *we have*

$$h_k(u) \geq i$$

<u>or</u> *for all* $i \in A^*_{>\sup(u)} \cup \{j^*\}$ *we have*

$$h_k(u) < i.$$

1.2 A *Remark.* (1) We could have also related $f_{k_1}(u)$, $f_{k_2}(u)$ for various k_1, k_2, this would not have influenced the bounds.

PROOF: Standard ramification. For $B \subseteq A$ we define an equivalence relation E_B on $A_{>\sup(B)}$ as follows. We let:
$i_0 \, E_B \, i_1$ <u>iff</u> $i_0, i_1 \in A_{>\sup(B)}$ and for every $u, v \in [B]^{n^*-1}$, $d \in \mathrm{Rang}(g)$, $w \in [B]^{n^*}$ and $k < k(*)$ the truth value of the following is the same for $\ell \in \{0, 1\}$:

(α) $f_k(u \cup \{i_\ell\}) = f_k(v \cup \{i_\ell\})$ if u, v are neighbors
(β) $f_k(u \cup \{i_\ell\}) = f_k(w)$ if $u = w \backslash \{\max(w)\}$
(γ) $g(u \cup \{i_\ell\}) = d$
(δ) $h_k(u) \geq i_\ell$.

Clearly E_B is an equivalence relation and $E_\emptyset$ is the equality (as $n^* > 1$).
For $i \in A_{>\sup(B)}$ we let i/E_B denote the equivalence class of i via E_B.
Note that if $B \subseteq B^*$, then $i/E_{B^*} \subseteq i/E_B$.
We now define a tree T by defining by induction on $\ell \in \mathbb{N}$ objects $t_{\leq \ell}$, $\leq_\ell$ and $\langle A_i : i \in t_{\leq \ell} \rangle$ such that:

(a) $(t_{\leq \ell}, \leq_\ell)$ is a tree, $t_{\leq \ell}$ a subset of A, $\leq_\ell$ a partial order on t_ℓ such that for every $x \in t_{\leq \ell}$, $\{y : y \leq_\ell x\}$ is linearly ordered
(b) $t_{\leq \ell} \subseteq t_{\leq \ell+1}$ and $\leq_{\ell+1} \restriction t_{\leq \ell} = \leq_\ell$
(c) $t_{\leq 0} = \{\min(A)\}, A_{\min(A)} = A_{>\min(A)}$
(d) $t_{\leq(\ell+1)} \backslash t_{\leq \ell}$ is the $(\ell+1)$-th level of $(t_{\leq(\ell+1)}, \leq_{\ell+1})$
(e) if $i_0 <_\ell i_1 <_\ell \cdots <_\ell i_\ell \in t_{\leq \ell}$ (so $\{i_0, \ldots, i_\ell\}$ is a branch) then
  (α) $A_{i_\ell} = i_\ell / E_{\{i_0, \ldots, i_{\ell-1}\}}$
  (β) the set of immediate successors of i_ℓ in $(t_{\leq(\ell+1)}, \leq_{\ell+1})$ is $Y_{i_\ell} =: \{\min(j/E_{\{i_0, i_1, \ldots, i_\ell\}}) : j \in A_{i_\ell}$ but $j \neq i_\ell\}$.

This is straight. Let $t_\ell = t_{\leq \ell} \backslash \bigcup\limits_{m<\ell} t_{\leq m}$ and $T = \bigcup\limits_{\ell} t_{\leq \ell}$.
Note also that $i \leq_\ell j \Rightarrow i \leq j$ and that

$\otimes$ if we consider the definition of $E_{\{i : i \leq_\ell j\}}$ restricted just to $A_j \backslash \{j\}$ we may restrict ourselves: for clause (α) only to the $u, v \in [\{i : i \leq_\ell j\}]^{n^*-1}$ with $\max(u \cup v) = j$, and for clause (β) only to those $u \in [\{i : i \leq_\ell j\}]^{n^*-1}$, $w \in [\{i : i \leq_\ell j\}]^{n^*}$ with $\max(w) = j$.
For (γ) and (δ) we may assume $\max(u) = i_\ell$.

Now it is easy to see that

$(*)_3$ $A = \bigcup_\ell t_\ell$

$(*)_4$ if $j \in t_\ell$ then the number of immediate successors of j in $(t_{\leq \ell+1}, \leq_{\ell+1})$ (necessarily they are all in $t_{\ell+1}$) is at most

$$\left(2^{\binom{\ell}{n^*-1}(n^*-1)}\right)^{k(*)} \times \left(2^{\binom{\ell}{n^*-1}}\right)^{k(*)} \times t^{\binom{\ell}{n^*-2}} \times \left(\binom{\ell}{n^*-2} \cdot k(*) + 1\right).$$

[Why this inequality? The four terms in the product correspond to the four clauses $(\alpha), (\beta), (\gamma), (\delta)$ in the definition of E_B for the branch $B = \{i_0, \ldots, i_\ell = j\}$ of $(t_{\leq \ell}, \leq)$. The power $k(*)$ in the first two terms comes from dealing with f_k for each $k < k(*)$ and "2 to the power x" as we have x choices of yes/no. Now the first term comes from counting the possible $u \cup v$ (from clause (α)). At the first glance their number is $|[\{i_0, \ldots, i_\ell\}]^{n^*}|$ as being neighbors each with $n^* - 1$ elements they have together n^* elements, but by $\bigotimes$ we can restrict ourselves to the case $i_\ell \in u \cup v$, so we have to consider $|[\{i_0, \ldots, i_{\ell-1}\}]^{n^*-1}| = \binom{\ell}{n^*-1}$ sets $u \cup v$; then we have to choose $u \cup v \backslash (u \cap v)$ (as we do not need to distinguish between (u, v) and (v, u)). As u, v are neighbors we have $n^* - 1$ possible choices (as the two members of $(u \cup v) \backslash (u \cap v)$ are successive members of $u \cup v$ under the natural order).

For the second term, we should consider u, w as in clause (β), and so as $u = w \backslash \{\max(w)\}$ we know w gives all the information, and by $\bigotimes$ above $\max(w) = i_\ell$, so the number of possibilities is $\binom{\ell}{n^*-1}$.

For the third term we have a choice of one from $\leq t (= |\mathrm{Rang}(g)|)$ for each $u \in [\{i_0, \ldots, i_\ell\}]^{n^*-1}$, but again by $\bigotimes$, with $\max(u) = i_\ell$, so the number is $\binom{\ell}{n^*-2}$.

Lastly, in the fourth term the number of questions "$h_k(u) \geq i$" is again $\binom{\ell}{n^*-2} \cdot k(*)$, but by the properties of linear orders there are $\binom{\ell}{n^*-2} \cdot k(*) + 1$ possible answers. So $(*)_4$ really holds.]

Clearly (with $c_0 = k(*)/(n^* - 2)! + k(*)/(n^* - 1)!$)

$$(*)_5 \quad \left(2^{\binom{\ell}{n^*-1} \times n^*-1}\right)^{k(*)} \times \left(2^{\binom{\ell}{n^*-1}}\right)^{k(*)} \times t^{\binom{\ell}{n^*-2}} \times \left(\binom{\ell}{n^*-2} \cdot k(*) + 1\right)$$
$$\leq 2^{k(*)\ell^{n^*-1}/(n^*-2)!} \times 2^{k(*) \cdot \ell^{n^*-1}/(n^*-1)!} \times 2^{\log(t)\ell^{n^*-2}/(n^*-2)!}$$
$$\times \ell^{n^*-2} \cdot k(*)/(n^* - 2)! \leq 2^{c_0 \ell^{n^*-1}(1+\varepsilon)}.$$

(any positive ε, for ℓ large enough; actually we can replace ε by e.g. $1/\ell^{1-\varepsilon}$, $\varepsilon > 0$). So (for some constant c_0^2)

$$(*)_6 \quad |t_{\ell+1}| \le c_0^2 \prod_{p=1}^{\ell} 2^{(1+\varepsilon)c_0 p^{n^*-1}} = c_0^2 \cdot 2^{(1+\varepsilon)c_0 \sum_{p=1}^{\ell} p^{n^*-1}}$$

$$\le c_0^2 \cdot 2^{(1+\varepsilon)c_0(\ell+1)^{n^*}/n^*} = c_0^2 \cdot 2^{(1+\varepsilon)c_1^0(\ell+1)^{n^*}}.$$

But

$\bigoplus$ if $a_p \ge 0, a_p \le a_{p+1}$ and $p \ge \ell^* \Rightarrow 2a_p \le a_{p+1}$ then

$$\sum_{p=0}^{\ell} a_p \le 2a_\ell + \sum_{p \le \ell^*} a_p$$

hence (possibly increasing ε, which means for $(*)_5$ using large ℓ)

$$(*)_7 \quad |t_{\le(\ell+1)}| \le c_0^3 + \sum_{p=0}^{\ell+1} c_0^2 \cdot 2^{(1+\varepsilon)c_1^0(p+1)^{n^*}} \le c_0^4 \cdot 2^{(1+\varepsilon)c_1^0(\ell+1)^{n^*}+1} \le 2^{(1+\varepsilon)c_1(\ell+1)^{n^*}}.$$

So (increasing ε slightly)

$$(*)_8 \quad |t_{\le m^*}| \le m < |A|$$

so there is a $j^* \in t_{m^*+1}$. Let $A^* = \{i : i <_{m^*+1} j^*\}$ (so $|A^*| = m^* + 1$), then A^*, j^* are as required (actually we could have retained c_0 instead c_1). $\square_{1.2}$

1.3 Claim. *Assume*

$(*)_9$ (a) $n^\otimes \ge n^* \ge 1$, $k(*) > 0$

(b) *we have the function* $m(-)$ *satisfying* $m(n+1) \ge 2^{(1+\varepsilon)c_1 m(n)^{n+1}}$ *for* $n \in [n^*, n^\otimes)$, c_1 *from 1.2*

(c) $t > 0$ *and* $m(n^*)$ *is large enough relative to* $k(*), n^*, c_1, 1/\varepsilon$.

$(*)_{10}$ $A \subseteq \mathbb{N}$, $|A| \ge m(n^\otimes) + 1$, g *is a function with domain* $[A]^{n^\otimes}$ *and range with* $\le t$ *members;* f_k *is a function with domain* $[A]^{n^\otimes}$ *(for* $k < k(*)$*), and for simplicity* $\mathcal{P}(\{0, 1, \ldots, n^\otimes - 1\}) \cap \mathrm{Rang}(f_k) = \emptyset$.

Then we can find $A' \in [A]^{m(n^*)+1}$ *and* $j_\ell^* \in A$ *for* $\ell \in [n^*, n^\otimes)$ *satisfying* $A' < j_{n^*}^* < j_{n^*+1}^* < \ldots$, *and functions* g', g_k, h_k $(k < k(*))$ *with domain* $[A']^{n^*}$ *such that (letting* $w^* = \{j_\ell^* : \ell \in [n^*, n^\otimes)\}$*):*

$(*)_{11}$ *for all* $u \in [A']^{n^*}$

(a) *for* $w \in [A'_{>\sup(u)}]^{n^\otimes - n^*}$ *we have* $g(u \cup w) = g'(u) = g(u \cup w^*)$

(b) *for* $k < k(*)$ *we have* $h_k(u) \in \mathbb{N}$ *and* $g_k(u) \in \{v : v \subseteq (n^*, n^\otimes)\}$

(c) *if* $w_1, w_2 \in [A'_{>\sup(u)}]^{n^\otimes - n^*}$ *and* $k < k(*)$ *and:* (*note:* $|u| = n^*$)
$\{i \in w_1 : |u| + |i \cap w_1| \in g_k(u)\} = \{i \in w_2 : |u| + |i \cap w_2| \in g_k(u)\}$
and $[\min(w_1 \cup w_2) < h_k(u) \Rightarrow \min(w_1) = \min(w_2)]$ *then*
$f_k(u \cup w_1) = f_k(u \cup w_2)$.

(d) *Assume* $k < k(*)$, $w_1 \cup w_2 \cup \{i, j\} \subseteq A'$, $u < w_1 < i < j < w_2$ *and* $|w_1 \cup w_2| = n^\otimes - n^* - 1$:

(i) *if* $w_1 \neq \emptyset$ *then*

$$f_k(u \cup w_1 \cup \{i\} \cup w_2) = f_k(u \cup w_1 \cup \{j\} \cup w_2) \Leftrightarrow |u \cup w_1| \notin g_k(u)$$

(ii) *if* $w_1 = \emptyset$ *then*

$$f_k(u \cup \{i\} \cup w_2) = f_k(u \cup \{j\} \cup w_2) \Leftrightarrow h_k(u) \leq i.$$

(e) *for* $k < k(*)$ *and neighbors* $u_0, u_1 \in [A']^{n^*}$ *and* $w \in [A'_{>\max(u_0 \cup u_1)}]^{n^\otimes - n^*}$ *we have:*

$$f_k(u_0 \cup w) = f_k(u_1 \cup w) \text{ iff } f_k(u_0 \cup w^*) = f_k(u_1 \cup w^*).$$

Remark.. (1) Note particularly clause (d). So $g_k(u)$ is intended to be like the v in 1.1, only fixing an initial segment of both $\{i_\ell : \ell < n^\otimes\}$ and $\{j_\ell : \ell < n^\otimes\}$ as u. But whereas the equality demand in clause (d) is as expected, the non-equality demand is weaker: only for neighbors.

(2) Note that we can in some clauses above replace A' by $A' \cup w^*$.

Proof: We prove this by induction on $n^\otimes - n^*$. If it is zero, the conclusion is trivial.

Use the induction hypothesis with $n^\otimes$, $n^* + 1$, f_k, $(k < k(*))$, g now standing for $n^\otimes$, n^*, f_k, $(k < k(*))$, g in the induction hypothesis. We get $A' \in [A]^{m(n^*+1)+1}$ and functions g', g_k, h_k (for $k < k(*)$) and j^*_ℓ for $\ell \in [n^* + 1, n^\otimes)$ satisfying $(*)_{11}$ of Claim 1.3. Now we apply 1.2 to $n^* + 1$ and $m = m(n^* + 1), A', g^\otimes, f^\otimes_k, h^\otimes_k$ $(k < k(*))$ where we define the function $g^\otimes$ with domain $[A']^{n^*+1}$ by $g^\otimes(u) = \langle g'(u), g_k(u) : k < k(*)\rangle, h^\otimes_k = h_k$ and the function $f^\otimes_k$ with domain $[A']^{n^*+1}$ is defined by

$$f^\otimes_k(u) = f_k(u \cup \{j^*_\ell : \ell \in [n^* + 1, n^\otimes)\}).$$

We get there $A^* \in [A']^{m(n^*)+1}$ and $j^* \in (A')_{>\sup A^*}$. Let $j_{n^*} =: j^*$. Now we have to define h_k with domain $[A^*]^{n^*}$ (for $k < k(*)$). For $u \in [A^*]^{n^*}$ let

$$B^k_u =: \{i \in A^*_{>\sup(u)} : f^\otimes_k(u \cup \{i\}) \neq f^\otimes_k(u \cup \{j^*\})\}.$$

By clause (β) of Claim 1.2, B^k_i is an initial segment of $A^*_{>\sup(u)}$. Let $h_k(u) = \max(B^k_u) + 1$.

Lastly, for $u \in [A^*]^{n^*}$ we have to define $g_k(u)$. By 1.2 (δ), the answer to "$h^{\otimes}_k(u \cup \{j\}) < j_{n^*}$" does not depend on $j \in A^*_{>\sup u}$. Let $g^{\otimes}_k$ be the "old" g_k (with domain $[A']^{n^*+1}$) and let

$$g_k(u) = \begin{cases} g^{\otimes}_k(u \cup \{j_{n^*}\}) & \text{if } h^{\otimes}_k(u \cup \{j\}) < j_{n^*} \\ g^{\otimes}_k(n \cup \{j_{n^*}\}) \cup \{j_{n^*}\} & \text{otherwise.} \end{cases}$$

Now A^*, g_k, h_k, j^*_n, $j^*_{n^*+1}, \dots$ are as required. $\square_{1.3}$

1.4 Claim. (1) *Assume* $m(1) \geq (k(*) \cdot m(0))^{k(*)+1}$ *and* $A' \subseteq \mathbb{N}, |A'| \geq m(1)$ *and for* $k < k(*)$, h_k *is a function from* A' *into* $\mathbb{N}$, $h_k(i) \geq i$.

Then we can find $A'' \subseteq A'$, $|A''| \geq m(0)$ *such that*

$(*)_{12}$ *for each* $k < k(*)$ *we have:*

$$\begin{array}{ll} \textit{either} & (\forall\, i, j \in A'')[i < j \Rightarrow h_k(i) \geq j] \\ \textit{or} & (\forall\, i, j \in A'')[i < j \Rightarrow h_k(i) < j]. \end{array}$$

(2) *If* $m(1) > d^k m(0)^{2^{k(*)}}$, $A \subseteq \mathbb{N}$, $|A| > m(1)$, g_k *is a function from* A *to* $\{1, \dots, d\}$, *and* f_k *is a function from* A *to* $\mathbb{N}$ *for* $k < k(*)$ *then we can find* $A' \subseteq A$, $|A'| > m(0)$ *such that:*

$\bigotimes$ *for each* k, $f_k \restriction A'$ *is constant or one to one and* $g_k \restriction A'$ *is constant.*

Proof: (1) We can find $A_1 \subseteq A'$, $|A_1| \geq m(1)/k(*)^{k(*)}$ such that for all $i, j \in A_1$,

$$\ell, k < k(*) \Rightarrow [h_\ell(i) \leq h_k(i) \equiv h_\ell(j) \leq h_k(j)].$$

So without loss of generality

$(*)$ $\ell < k < k(*)$ & $i \in A_1 \Rightarrow h_\ell(i) \leq h_k(i)$.

By renaming we can assume $A_1 = \{1, 2, \dots, m(0)^{k(*)+1}\}$.
Now if for some $\ell, 0 < \ell \leq m(0)^{k(*)+1} - m(0)$, and $(\forall\,\alpha)\Big(\alpha \in [\ell, \ell+m(0)) \Rightarrow h_0(\alpha) \geq \ell+m(0)\Big)$ then $A'' = [\ell, \ell+m(0))$ is as required for all h_ℓ by $(*)$.
If not, then we can find $\alpha_\ell \in [1, m(0)^{k(*)+1})$ for $\ell = 1, \dots, m(0)^{k(*)}$, strictly increasing with ℓ such that $h_0(\alpha_\ell) < \alpha_{\ell+1}$. We repeat the argument for h_1, etc.
(2) Also easy. $\square_{1.4}$

Remark.. We can use $m(1) > k(*)! \cdot m(0)^{k(*)+1}$ instead. The only point is the choice of A.

1.5 Claim. *Assume we have the assumptions of* 1.3. *If we first apply* 1.3 *getting* A' *and then apply* 1.4 *to get* $A'' \subseteq A'$ *such that for each* k *and* $u \in [A'']^{n^*}$ *either* $h_k(u) \leq \min\{\ell \in A'' : u < \ell\}$ *or* $h_k(u) > \max(A'')$ (*we assume now* $n^* = 1$ *so* $u = \{j\}$), *and in addition*

$(*)_{13}$ $m^{2n^{\otimes}-n^*} \cdot (2n^{\otimes} - n^*)\binom{2(n^{\otimes}-n^*)-1}{n^{\otimes}-n^*} \cdot k(*) \leq |A''|$

then there is $A^* \in [A'']^m$ *such that (in addition to* $(*)_{11}$(a)–(e) $+(*)_{12}$) *we have*

$(*)_{14}$ *for all* $u \in [A^*]^{n^*}$

(f) *if* $w_1, w_2 \in [A^*_{>\sup(u)}]^{n^{\otimes}-n^*}, k < k(*)$ *then*

$$f_k(u \cup w_1) = f_k(u \cup w_2) \Leftrightarrow \{i \in w_1 : |i \cap w_1| + |u| \in g_k(u)\} = \{i \in w_2 : |i \cap w_2| + |u| \in g_k(u)\}.$$

Remark.. Here we are rectifying the gap between the equality $((*)_{11}(\mathrm{d}))$ and the inequality $((*)_{11}(\mathrm{e}))$ demand.

Proof: First note that

$(*)_{15}$ for all $u \in [A'']^{n^*}$ the implication $\Leftarrow$ holds.
[why? just use clause (c) of $(*)_{11}$ of Claim 1.3].

So we are left with proving $\Rightarrow$.

Choose randomly m members of A''. We shall prove that the probability that the set they form has exactly m members and satisfies clause (f), is positive. This suffices. Let us explain. We fix n^* among these elements and call the set they form u.

In clause $(*)_{12}$ for $\ell = 1, 2$ we let $v_\ell =: \{i \in w_\ell : |u| + |i \cap w_\ell| \in g_k(u)\}$.
By $(*)_{15}$ the problem is that $\Rightarrow$ may fail.

Let $x_1, \ldots, x_m$ be random variables on A''. The probability that $\bigvee_{i \neq j} x_i = x_j$ is $\leq \binom{m}{2} \cdot \frac{1}{|A''|}$.

Now for $k < k(*)^1$, $u \in [\{1, \ldots, m\}]^{n^*}$, $w_1, w_2 \in [\{1, \ldots, m\} \backslash u]^{n^{\otimes}-n^*}$, $v_1 \subseteq w_1, v_2 \subseteq w_2$ defined as above, and a possible linear order $<^*$ on $u^* = u \cup w_1 \cup w_2$, we shall give an upper bound for the probability that

$$\bigwedge_{\ell_1, \ell_2 \in u^*} (\ell_1 <^* \ell_2 \Leftrightarrow x_{\ell_1} < x_{\ell_2})$$

and they form a counterexample to clause (f) (of Claim 1.5). So in particular $u < w_1, u < w_2$.

Choose $\ell \in v_1 \backslash v_2$ (as $v_1 \neq v_2$ and $|v_1| = |g_k(u)| = |v_2|$ it exists). We can first draw x_j for $j \neq \ell$. Now we know $f_k(u \cup w_2)$; note: we may not know w_2 as possibly $\ell \in w_2$, but as $\ell \notin v_2$, by the choice of A' it is not necessary to know w_2. Now there is at most one bad choice of x_ℓ (the others are good (inequality) or irrelevant ($<^*$ is not right) by (d) + (e)) so the probability of this is $\leq \frac{1}{|A'|}$. So if we fix the set $u^* = u \cup w_1 \cup w_2$ and concentrate on the case $|u^*| = 2n^\otimes - n^*$, we have $2n^\otimes - n^*$ possibilities to choose $\ell \in u^*$ and then having to choose x_i for $i \neq \ell$, we know u and have $\leq \binom{2(n^\otimes - n^*)-1}{n^\otimes - n^*}$ ways to choose w_2, so the probability of failure is $\leq (2n^\otimes - n)\binom{2(n^\otimes - n^*)-1}{n^\otimes - n^*} \cdot \frac{1}{|A''|}$.
So the probability that some failure occurs is at most (the cases $|u^*| < 2n^\otimes - n^*$ and $x_1 = x_2$ are swallowed when m is not too small)

$$m^{2n^\otimes - n^*} \cdot k(*)(2n^\otimes - n)\binom{2(n^\otimes - n^*) - 1}{n^\otimes - n^*} \cdot \frac{1}{|A''|}.$$

Now by assumption $(*)_{13}$ this probability is < 1 so the conclusion is clear. $\square_{1.5}$

Before we state and prove the last fact, which finishes the proof of the theorem, we remind the reader of the following observation. The proof is easily obtained by induction on ℓ.

1.6 Observation. (1) *$\beth_\ell(kx) \geq k\beth_\ell(x)$ when $x, k \geq 2$ and $\ell \geq 1$.*
(2) *$\beth_\ell(kx) \geq (\beth_\ell(x))^k$ when $x \geq 2$, $k \geq 2$ and $\ell \geq 1$.*

1.7 Fact. *Assume that $n^\otimes, n^*, m(n^*), k(*), \varepsilon, t$ and c_1 are as in $(*)_9$ (a) and (c).*

Let us define

$$c_2 = \text{Max}\{(1+\varepsilon)c_1, 2\}$$
$$c_3 = n^\otimes \times (c_2)^2 \quad \text{(in fact } n^\otimes \times c_2 \text{ suffices)}$$

and the function $m(-)$ as follows: for $n \in (n^*, n^\otimes]$ by

$$m(n) = \beth_{n-n^*}(m^{n^*+1} c_3^{n-n^*})$$

where

$$m(n^*) = m.$$

Then $(*)_9$ (b) holds.

Proof: We need to check that for $n \in [n^*, n^\otimes)$

$$m(n+1) \geq 2^{(1+\varepsilon)c_1 m(n)^{n+1}}$$

or equivalently

$$\log_2(m(n+1)) \geq (1+\varepsilon)c_1\, m(n)^{n+1}$$

so it is enough that

$$\log_2(m(n+1)) \geq c_2\, m(n)^{n+1}$$

i.e.

$$\log_2(\beth_{n+1-n^*}(m^{n^*+1}c_3^{n+1-n^*})) \geq c_2\, m(n)^{n+1}$$

i.e., when $n > n^*$

$$\beth_{n-n^*}(c_3(c_3^{n-n^*})m^{n^*+1}) \geq c_2(\beth_{n-n^*}(c_3^{n-n^*}\, m^{n^*+1}))^{n+1}.$$

It suffices by the above observation that

$$\beth_{n-n^*}(c_3 \cdot c_3^{n-n^*}\, m^{n^*+1}) \geq \beth_{n-n^*}(c_2(n+1)c_3^{n-n^*}\, m^{n^*+1}),$$

which is true by the definition of c_3 when $n > n^*$.

For $n = n^*$ we need that

$$m(n^*+1) \geq 2^{(1+\varepsilon)c_1 m^{n^*+1}}$$

i.e.

$$2^{m^{n^*+1}c_3} \geq 2^{(1+\varepsilon)c_1 m^{n^*+1}},$$

which is true as $c_3 \geq c_2 \geq (1+\varepsilon)c_1$. $\square_{1.7}$

1.8 Proof of Lemma 1.1. So m, n, ε are given. Let

(a) $n^* = 1$, $n^{\otimes} = n$, $k(*) = 1$, $t = 1$, c_1 as in 1.2, and c_2, c_3 as in 1.7
(b) $m_0 = m$

$$\begin{aligned} m_1 &= k(*) \cdot (2n^{\otimes} - n^*) \cdot \binom{2(n^{\otimes} - n^*) - 1}{n^{\otimes} - n^*}(m_0)^{2n^{\otimes} - n^*} \\ &= (2n-1)\binom{2n-3}{n-1}m^{2n-1} \end{aligned}$$

$$\begin{aligned} m_2 &= (k(*)m_1)^{k(*)+1} = (m_1)^2 \\ m_3 &= (m_1)^2 \cdot 2^{k(*)} = 2(m_1)^2 \end{aligned}$$

(c) we define function $m(-)$ with domain $[n^*, n^{\otimes}]$:

$$\text{for } \ell = 1 \text{ we let } m(1) = m_3$$

$$\text{for } \ell > 1 \text{ we let } m(\ell) = \beth_{\ell-1}(c_3^{\ell-1} \cdot (m_3)^2).$$

So we are given $m^{\otimes} > m(n)$. In Claim 1.3 from the assumption $(*)_9$, clauses (a), (c) hold and clause (b) holds by Fact 1.7. Also assumption $(*)_{10}$ of 1.3 holds (with $\{1, \dots, m^{\otimes}\}$ standing for A, f_0 the given function f (in 1.1), and g constantly zero).

So there are $A \in [\{1, \dots, m^{\otimes}\}]^{m(1)+1}$, g', g_0, h_0 satisfying the conclusion of 1.3 i.e. $(*)_{11}$. A here stands for A' in 1.3. Note $|A| = m_3 + 1$. Now apply 1.4 (2) with A, m_3, m_2, f_0, g_0, 1 here standing for A, $m(1)$, $m(0)$, f_0, g_0, $k(*)$ there and get $A' \in [A]^{m_2+1}$. Next we apply Claim 1.4 (1) with A', m_2, m_1, h_0, 1 here standing for A', $m(1)$, $m(0)$, h_0, $k(*)$ there and get $A'' \in [A']^{m_1}$ satisfying the conclusion of 1.4 (1) i.e. $(*)_{12}$. Lastly apply Claim 1.5 and get $A^* \in [A'']^{m_0} = [A'']^m$ satisfying the conclusion of 1.5; i.e. $(*)_{14}$. Now A^* is as required.

1.9 *Remark.* We could have applied 1.5 in each stage, or just for $n = 3$, this saves, somewhat.

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