On half-completion and bicompletion of quasi-metric spaces

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Abstract. We characterize the quasi-metric spaces which have a quasi-metric half-completion and deduce that each paracompact co-stable quasi-metric space having a quasi-metric half-completion is metrizable. We also characterize the quasi-metric spaces whose bicompletion is quasi-metric and it is shown that the bicompletion of each quasi-metric compatible with a quasi-metrizable space X is quasi-metric if and only if X is finite.

Keywords: quasi-metric, quasi-uniform, half-completion, bicompletion, uniformly weakly regular

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1. Introduction and preliminaries

Terms and concepts which are not defined may be found in [FL]. Paracompact spaces are assumed to be regular. If A is a subset of a set X and T is a topology on X, then $TclA$ will denote the closure of A in the topological space (X, T). The letters $\mathbb{N}$ and $\mathbb{R}$ will denote the set of positive integer numbers and the set of real numbers, respectively.

A quasi-pseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, x) = 0$, and

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If d satisfies the additional condition

(iii) $d(x, y) = 0 \Leftrightarrow x = y$,

then d is called a quasi-metric on X.

The conjugate of a quasi-(pseudo)metric d on X is the quasi-(pseudo)metric d^{-1} given by $d^{-1}(x, y) = d(x, y)$. By d^* we denote the (pseudo)metric given by $d^*(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$.

Each quasi-metric d on X generates a topology $T(d)$ on X which has as a base the family of d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

A topological space (X, T) is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X compatible with T, where d is compatible with T provided

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that $T = T(d)$. (X, T) is said to be strongly quasi-metrizable ([St], [Kü1]) if there is a quasi-metric d on X compatible with T such that $T(d) \subseteq T(d^{-1})$.

According to [RSV] a quasi-pseudometric space (X, d) is called d-sequentially complete if each Cauchy sequence in (X, d^*) is $T(d)$-convergent to a point in X and it is called left K-sequentially complete if each left K-Cauchy sequence in (X, d) is $T(d)$-convergent to a point in X, where a sequence $\langle x_n \rangle$ in (X, d) is called left K-Cauchy if for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $k \leq n \leq m$.

A quasi-pseudometric space (X, d) is called bicomplete ([RS], [KRS]) if the pseudometric space (X, d^*) is complete. While every bicomplete quasi-pseudometric space is d-sequentially complete, it is easy to obtain examples of d-sequentially complete non bicomplete quasi-metric spaces.

The Sorgenfrey line, the Kofner plane and the Pixley-Roy space on $\mathbb{R}$ are relevant examples on nonmetrizable spaces which admit a compatible bicomplete quasi-metric.

If (X, d) is a quasi-metric space, we say that (Y, q) is a quasi-metric d-sequential completion of (X, d) if (Y, q) is a q-sequentially complete quasi-metric space such that (X, d) is isometric to a $T(q)$-dense subspace of (Y, q) ([RSV]). Similarly we define the notion of a quasi-metric left K-sequential completion of (X, d). The example given in [RG] of a Hausdorff quasi-metric space which does not have a quasi-metric d-sequential completion, suggests the question of characterizing those quasi-metric spaces which admit a quasi-metric d-sequential completion. In Proposition 1 of this paper we shall give an answer to this question.

According to [Sa], [FL], we say that (Y, q) is a bicompletion of the quasi-pseudometric space (X, d) if (Y, q) is a bicomplete quasi-pseudometric space such that (X, d) is isometric to a $T(q^*)$-dense subspace of (Y, q). If (X, d) is a quasi-metric space and its bicompletion (Y, q) is also a quasi-metric space, we shall say that (Y, q) is a quasi-metric bicompletion of (X, d). Salbany showed in [Sa] that each T_0 quasi-pseudometric space has an (up to isometry) unique T_0 bicompletion. In Proposition 4 of this paper we shall characterize the quasi-metric spaces which admit a quasi-metric bicompletion. On the other hand, it is proved in [SR] that each quasi-metric compatible with a quasi-metrizable space (X, T) admits a quasi-metric d-sequential completion if and only if (X, T) is compact. The corresponding result to quasi-metric bicompletions will be stated in Proposition 5, where we shall show that the bicompletion of each quasi-metric compatible with a quasi-metrizable space X is quasi-metric if and only if X is a finite set.

According to [FL], if $(X, \mathcal{U})$ is a quasi-uniform space, we shall denote by $\mathcal{U}^*$ the coarsest uniformity on X which is finer than $\mathcal{U}$ (i.e. $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$).

Let us recall that a quasi-uniform space $(X, \mathcal{U})$ is half-complete provided that each Cauchy filter on the uniform space $(X, \mathcal{U}^*)$ is $T(\mathcal{U})$-convergent to a point in X ([De1]). $(X, \mathcal{U})$ is called bicomplete ([Sa], [FL]) if each Cauchy filter on $(X, \mathcal{U}^*)$ is $T(\mathcal{U}^*)$-convergent to a point in X.

Let $(X, \mathcal{U})$ be a T_1 quasi-uniform space. A T_1 quasi-uniform half-completion of $(X, \mathcal{U})$ is a half-complete T_1 quasi-uniform space $(Y, \mathcal{V})$ in which $(X, \mathcal{U})$ can be quasi-uniformly embedded as a $T(\mathcal{V})$-dense subspace ([Ro3]).

Now let (X,d) be a quasi-pseudometric space and let $\mathcal{U}(d)$ be the quasi-uniformity on X generated by d (i.e. $\mathcal{U}(d)$ is the quasi-uniformity on X that has as a base the family of all sets of the form $\{(x,y)\in X\times X : d(x,y)<2^{-n}\}$). Then we say that (X,d) is half-complete if the quasi-uniform space $(X,\mathcal{U}(d))$ is half-complete. If (X,d) is a quasi-metric space, we say that (Y,q) is a quasi-metric half-completion of (X,d) if (Y,q) is a half-complete quasi-metric space such that (X,d) is isometric to a $T(q)$-dense subspace of (Y,q). It follows from [SR, Lemma 1] that a quasi-pseudometric space is d-sequentially complete if and only if it is half-complete. As an immediate consequence of this result we have the following

Lemma 1. *A quasi-metric space has a quasi-metric d-sequential completion if and only if it has a quasi-metric half-completion.*

We conclude this section with some notions of the theory of quasi-uniform spaces which will be used later on.

Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is called:

(i) $\mathcal{U}$-stable if for each $U\in\mathcal{U}$, $\bigcap\{U(F) : F\in\mathcal{F}\}\in\mathcal{F}$ ([Cs]);

(ii) D-Cauchy if there is a filter $\mathcal{G}$ on X such that $(\mathcal{G},\mathcal{F})\to 0$, where $(\mathcal{G},\mathcal{F})\to 0$ if for each $U\in\mathcal{U}$ there exist $F\in\mathcal{F}$ and $G\in\mathcal{G}$ such that $G\times F\subseteq U$ ([Do], [FH2]).

A quasi-uniform space $(X,\mathcal{U})$ is called co-stable ([DR]) if each D-Cauchy filter on $(X,\mathcal{U}^{-1})$ is $\mathcal{U}$-stable. A quasi-metric space (X,d) is said to be co-stable if $(X,\mathcal{U}(d))$ is a co-stable quasi-uniform space.

2. Quasi-metric spaces having a quasi-metric half-completion

Proposition 1. *For a quasi-metric space (X,d) the following conditions are equivalent:*

(1) *(X,d) has a quasi-metric half-completion;*

(2) *whenever $\langle x_n\rangle$ is a Cauchy sequence in the metric space (X,d^*) which is $T(d^{-1})$-convergent to a point $x\in X$, then $\langle x_n\rangle$ is $T(d)$-convergent to x;*

(3) *the quasi-uniform space $(X,\mathcal{U}(d))$ has a T_1 quasi-uniform half-completion.*

Proof: (1) $\Rightarrow$ (3). Obvious.

(2) $\Rightarrow$ (1). In order to prove this implication we shall use a construction due to Künzi [Kü2, Lemma 7]:

Let $\mathcal{A}=\{x : x \text{ is a non } T(d)\text{-convergent Cauchy sequence in } (X,d^*)\}$ and let $Y=X\cup\mathcal{A}$. Given $x=\langle x_n\rangle\in\mathcal{A}$ there is a strictly increasing sequence $\langle j(n)\rangle$ of natural numbers such that for each $n\in\mathbb{N}$, $d(x_k,x_m)<2^{-n}$ whenever $k,m\geq j(n)$. Put $s(x)=x_{j(1)}$, $S_1(x)=B_d(x_{j(1)},2^{-1})$ and $S_n(x)=\{x_k : k\geq j(n)\}$ for $n>1$.

Now define for each $x, y \in Y$,

$$q(x,y) = \begin{cases} d(x,y) & \text{if } x, y \in X, \\ d(s(x), s(y)) + 2 & \text{if } x, y \in \mathcal{A},\ x \neq y, \\ 0 & \text{if } x, y \in \mathcal{A},\ x = y, \\ d(x, s(y)) + 3 & \text{if } x \in X,\ y \in \mathcal{A}, \\ \inf\{\max\{d(S_n(x), y), 1/n\} : n \in \mathbb{N}\} & \text{if } x \in \mathcal{A},\ y \in X. \end{cases}$$

According to [Kü2, Lemma 7], q is a quasi-pseudometric on Y such that $B_q(x, 2^{-m}) = (\bigcup\{B_d(a, 2^{-m}) : a \in S_{2^{m+1}}(x)\}) \cup \{x\}$ for all $x \in \mathcal{A}$ and all $m \in \mathbb{N}$. Therefore X is dense in $(Y, T(q))$. Now we show that q is actually a quasi-metric on Y: if $q(x,y) = 0$ for $x \in \mathcal{A}$ and $y \in X$, then $d(a_m, y) \to 0$ for some subsequence $\langle a_m \rangle$ of the Cauchy sequence in (X, d^*), $x = \langle x_n \rangle$. Hence, $d(y, a_m) \to 0$, a contradiction since $x \in \mathcal{A}$.

We finally prove that (Y, q) is q-sequentially complete. Let $\langle y_n \rangle$ be a Cauchy sequence in (Y, q^*). Then we can assume without loss of generality that $y_n \in X$ for all $n \in \mathbb{N}$ because $q(x,y) \geq 2$ for $x, y \in \mathcal{A}$, $x \neq y$. Thus $\langle y_n \rangle$ is a Cauchy sequence in (X, d^*). Suppose that $y = \langle y_n \rangle \in \mathcal{A}$. Since $S_{2^{m+1}}(y) \subseteq B_q(y, 2^{-m})$ for all $m \in \mathbb{N}$, we conclude that $q(y, y_n) \to 0$.

(3) $\Rightarrow$ (2). This implication is an immediate consequence of [Ro3, Proposition 7] which establishes that a T_1 quasi-uniform space $(X, \mathcal{U})$ has a T_1 quasi-uniform half-completion if and only if whenever $\mathcal{F}$ is a Cauchy filter on the uniform space $(X, \mathcal{U}^*)$ which is $T(\mathcal{U}^{-1})$-convergent to a point $x \in X$, then $\mathcal{F}$ is $T(\mathcal{U})$-convergent to x. □

Corollary 1. *Each θ-refinable co-stable quasi-metric space (X, d) admitting a quasi-metric half-completion is strongly quasi-metrizable.*

Proof: By Proposition 1, [Ro3, Corollary 7.2] and [Ro2, Corollary 2], (X, d) has a quasi-metric left K-sequential completion and, thus, $(X, T(d))$ s a strongly quasi-metrizable space ([Ro1, Corollary 5.1]). □

Corollary 2. *Each paracompact co-stable quasi-metric space (X, d) admitting a quasi-metric half-completion is metrizable.*

Proof: By Corollary 1, $(X, T(d))$ is strongly quasi-metrizable and, hence, developable. The result follows from the famous Bing's metrization theorem that every paracompact developable space is metrizable. □

Example 1. Let $X = \mathbb{R}$ and let d be the quasi-metric defined on X by $d(x,y) = \min\{1, |x - y|\}$ if x is rational, $d(x,y) = 1$ if $x \neq y$ and x is irrational, and $d(x,x) = 0$. Then $T(d)$ is the Michael line on $\mathbb{R}$. It is well known that $(\mathbb{R}, T(d))$ is a paracompact nonmetrizable space and it is shown in [DR] that $(\mathbb{R}, d)$ is a co-stable quasi-metric space. It follows from Corollary 2 that $(\mathbb{R}, d)$ does not admit a quasi-metric half-completion. Note, however, that d^{-1} is a left K-sequentially complete quasi-metric.

The notion of a uniformly regular quasi-uniform space plays a crucial role in the study of symmetry properties and completeness in quasi-uniform spaces (see, for instance, [De1], [De2], [De3], [FH1], [FH2], [KMRV], etc.). A quasi-uniform space $(X,\mathcal{U})$ is uniformly regular ([Cs]) provided that for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for all $x \in X$, $T(\mathcal{U})\,\mathrm{cl}\, V(x) \subseteq U(x)$.

If $(X,\mathcal{U})$ is a uniformly regular quasi-uniform space, then $(X, T(\mathcal{U}))$ is a regular topological space. The profusion of interesting examples of nonregular topological spaces suggests the following generalization of uniform regularity:

A quasi-uniform space $(X,\mathcal{U})$ is called *uniformly weakly regular* if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for all $x \in X$, $T(\mathcal{U})\,\mathrm{cl}\, V^*(x) \subseteq U(x)$. (As usual V^* denotes the entourage of $\mathcal{U}^*$, $V \cap V^{-1}$.)

A quasi-pseudometric space (X,d) is called *uniformly weakly regular* if the quasi-uniform space $(X,\mathcal{U}(d))$ is uniformly weakly regular.

It is easily seen that if $(X,\mathcal{U})$ is a uniformly weakly regular quasi-uniform space, then $(X, T(\mathcal{U}))$ is a R_0 topological space.

Example 2. Let $X = \mathbb{N}$ and let d be the quasi-metric defined on X by $d(n,m) = 1/m$ if $n < m$, $d(n,m) = 1$ if $n > m$ and $d(n,n) = 0$ for all $n \in \mathbb{N}$. Then $T(d)$ is the cofinite topology on $\mathbb{N}$ which is not regular. However $(\mathbb{N}, d)$ is uniformly weakly regular. (Note that $T(d^{-1})$ is the discrete topology on $\mathbb{N}$ and, hence, $(\mathbb{N},\mathcal{U}(d^{-1}))$ is uniformly regular.)

In [De1] Deák proved that each uniformly regular half-complete quasi-uniform space is bicomplete. Our next proposition generalizes this result to uniformly weakly regular spaces.

Proposition 2. *Each uniformly weakly regular half-complete quasi-uniform space is bicomplete.*

Proof: Let $(X,\mathcal{U})$ be a uniformly weakly regular half-complete quasi-uniform space. Let $\mathcal{F}$ be a Cauchy filter on the uniform space $(X,\mathcal{U}^*)$. Then $\mathcal{F}$ is $T(\mathcal{U})$-convergent to a point $x \in X$. We shall show that $\mathcal{F}$ is $T(\mathcal{U}^*)$-convergent to x. To this end, it suffices to prove that for each $U \in \mathcal{U}$, $U^{-1}(x) \in \mathcal{F}$. Given $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $T(\mathcal{U})\,\mathrm{cl}\, V^*(y) \subseteq U(y)$ for all $y \in X$. On the other hand, there is $F \in \mathcal{F}$ such that $F \times F \subseteq V$, so that $V^*(y) \in \mathcal{F}$ for all $y \in F$. Then $W(x) \cap V^*(y) \neq \emptyset$ for all $W \in \mathcal{U}$ and all $y \in F$. Thus $x \in T(\mathcal{U})\,\mathrm{cl}\, V^*(y) \subseteq U(y)$ for all $y \in F$. We conclude that $F \subseteq U^{-1}(x)$. This completes the proof. □

Proposition 3. *Let $(X,\mathcal{U})$ be a T_1 quasi-uniform space such that $(X,\mathcal{U}^{-1})$ is uniformly weakly regular. Then $(X,\mathcal{U})$ has a T_1 quasi-uniform half-completion.*

Proof: Let $\mathcal{F}$ be a Cauchy filter on $(X,\mathcal{U}^*)$ which is $T(\mathcal{U}^{-1})$-convergent to a point $x \in X$. By [Ro3, Proposition 7] cited above, it suffices to show that $\mathcal{F}$ is $T(\mathcal{U})$-convergent to x. Let $u \in \mathcal{U}$. Then there is a $V \in \mathcal{U}$ such that $T(\mathcal{U}^{-1})\,\mathrm{cl}\, V^*(y) \subseteq U^{-1}(y)$ for all $y \in X$. On the other hand, there is $F \in \mathcal{F}$ such that $V^*(y) \in \mathcal{F}$ for all $y \in F$. Then $W^{-1}(x) \cap V^*(y) \neq \emptyset$ for all $W \in \mathcal{U}$ and all

$y \in F$. Thus $x \in T(\mathcal{U}^{-1}) \operatorname{cl} V^*(y) \subseteq U^{-1}(y)$. We have shown that $F \subseteq U(x)$. Consequently, $\mathcal{F}$ is $T(U)$-convergent to x. $\square$

From Propositions 1 and 3 and Corollary 2 we immediately deduce the following

Corollary 3. *Each paracompact co-stable quasi-metric space whose conjugate is uniformly weakly regular is metrizable.*

Remark 1. Consider the quasi-metric space $(\mathbb{R}, d)$ of Example 1. It follows from Corollary 3 that the conjugate quasi-metric space $(\mathbb{R}, d^{-1})$ is not uniformly weakly regular. On the other hand, it is well known that $(\mathbb{R}, d)$ is uniformly regular.

3. Quasi-metric spaces having a quasi-metric bicompletion

Proposition 4. *The bicompletion of a quasi-metric space (X, d) is quasi-metric if and only if whenever $\langle x_n \rangle$ and $\langle y_n \rangle$ are Cauchy sequences in (X, d^*) such that $d(x_n, y_n) \to 0$, then $d(y_n, x_n) \to 0$.*

Proof: Suppose that the bicompletion (Y, q) of (X, d) is quasi-metric and let $\langle x_n \rangle$ and $\langle y_n \rangle$ be Cauchy sequences in (X, d^*) such that $d(x_n, y_n) \to 0$. Then there exist $a, b \in Y$ such that $q^*(a, x_n) \to 0$ and $q^*(b, y_n) \to 0$. By the triangle inequality, $q(a, b) = 0$. Thus $a = b$. Since $d(y_n, x_n) \leq q(y_n, a) + q(a, x_n)$, it follows that $d(y_n, x_n) \to 0$.

Conversely, let Y be the set of Cauchy sequences in (X, d^*). For each $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ in Y, put $q(x, y) = \lim_n d(x_n, y_n)$. Then q is a bicomplete quasi-pseudometric on Y such that X is $T(q^*)$-dense in Y (see [Sa, Theorem 2.3, p. 45]). Now let $R = \{(x, y) \in Y \times Y : q^*(x, y) = 0\}$. Then R is an equivalence relation. For each pair $[x]$, $[y]$, in the quotient Y/R, define $p([x], [y]) = q(x, y)$. Then p is a bicomplete quasi-pseudometric on Y/R such that $p^*([x], [y]) = 0 \Leftrightarrow [x] = [y]$ (see [Sa, Proposition 1.3, p. 42]). Clearly, the map $e : X \to Y/R$ defined by $e(x) = [x]$ for all $x \in X$, is an isometry from (X, d) into $(Y/R, p)$ and $e(X)$ is $T(p^*)$-dense in Y/R. Hence, $(Y/R, p)$ is a T_0 bicompletion of (X, d). We finally show that $(Y/R, p)$ is a quasi-metric space. In fact, if $p([x], [y]) = 0$, then $q(x, y) = 0$, so that $d(x_n, y_n) \to 0$, where $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ are two Cauchy sequences in (X, d^*). We conclude that $d(y_n, x_n) \to 0$ and, thus, $q(y, x) = 0$. Therefore $p^*([x], [y]) = 0$, which shows that $[x] = [y]$. $\square$

The next example deals with some natural conjectures that one may consider in the light of the obtained results.

Example 3. Let $X = \{1/n : n \in \mathbb{N}\}$ and let d be the quasi-metric defined on X by $d(1/(2n+1), 1/2m) = 1$ for all $n, m \in \mathbb{N}$, and $d(x, y) = |x - y|$ otherwise. Then both (X, d) and (X, d^{-1}) have a quasi-metric half-completion but (X, d) has no quasi-metric bicompletion as Proposition 4 shows. Note also that both $T(d)$ and $T(d^{-1})$ are the discrete topology on X, so both $\mathcal{U}(d)$ and $\mathcal{U}(d^{-1})$ are uniformly regular.

Proposition 5. *The bicompletion of each quasi-metric compatible with a quasi-metrizable space (X,T) is quasi-metric if and only if X is a finite set.*

Proof: Suppose that (X,T) is an infinite quasi-metrizable space such that the bicompletion of each compatible quasi-metric is quasi-metric. By [SR, Theorem 2] (X,T) is a compact space, so that it is second countable. Therefore it admits a compatible totally bounded quasi-metric d ([FL, Proposition 2.7]). Let $\langle x_n \rangle$ be a sequence of distinct points of X. Then it has a subsequence $\langle y_n \rangle$ which is Cauchy in (X,d^*). Since (X,T) is compact, $\langle y_n \rangle$ has a cluster point a. Let e be the quasi-metric defined on X by $e(x,a)=1$ if $x \neq a$ and $e(x,y)=\min\{1,d(x,y)\}$ otherwise. Clearly $T(e)=T$. Let (Y,q) be the bicompletion of (X,e). Then $\langle y_n \rangle$ is a Cauchy sequence in (Y,q^*), so that $q^*(y,y_n) \to 0$ for some $y \in Y$. Since $q(a,y_n) \to 0$, $a=y$ because q is a quasi-metric. Therefore $e^*(a,y_n) \to 0$ which contradicts that $e(x,a)=1$ for $x \neq a$. Consequently X is a finite set. The converse follows from [KRS, Corollary of Theorem 2]. □

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