On the range of a Jordan *-derivation

Péter Battyányi

Abstract. In this paper, we examine some questions concerned with certain "skew" properties of the range of a Jordan *-derivation. In the first part we deal with the question, for example, when the range of a Jordan *-derivation is a complex subspace. The second part of this note treats a problem in relation to the range of a generalized Jordan *-derivation.

Keywords: Jordan *-derivation

Classification: Primary 47B47, 47D50

Let $\mathcal{R}$ be a *-ring. An additive mapping $J : \mathcal{R} \to \mathcal{R}$ is called a Jordan *-derivation if it satisfies

$$J(a^2) = aJ(a) + J(a)a^* \qquad (a \in \mathcal{R}).$$

Jordan *-derivations were introduced in the papers of Šemrl (e.g. [12], [13]). The reason for introducing these mappings was the fact that the problem of representing quadratic forms by sesquilinear ones is closely connected with the structure of Jordan *-derivations. In this subject we would refer to articles [14], [19].

The structure of Jordan *-derivations on standard operator algebras was described by Šemrl [15]. From his results it also follows that a Jordan *-derivation on the algebra $\mathcal{B}(H)$ of all bounded linear operators in a real or complex Hilbert space H is of the form

$$J(T) = TA - AT^* \qquad (T \in \mathcal{B}(H))$$

for some operator $A \in \mathcal{B}(H)$.

Up to now many interesting results have been obtained concerning the structure and the range of a Jordan *-derivation (cf. [2], [6]–[10]). The present paper also fits into this area of research. These results concerning Jordan *-derivations were considerably motivated by the extensive investigations in relation to the structure and properties (range, norm etc.) of derivations. We would only mention some better known articles on this vast subject ([1], [5], [16]–[18]).

The results derived from a paper of Molnár [6] related to the range of a Jordan *-derivation are very similar to those obtained with respect to derivations by Stampfli [16]. Indeed, at first glance these mappings may appear to be very similar, and one can suspect that they bear the same properties in many aspects. Sometimes this is really the case, but, however, in closer investigations the slight difference between the definitions of the two kind of mappings can lead to completely different situations.

The first part of the paper is devoted to a problem which is quite characteristic of the behaviour of the range of a Jordan *-derivation. Observe first that for every $A \in \mathcal{B}(H)$ the range of the Jordan *-derivation induced by the operator A, that is, the set

$$\mathcal{R}_A = \{TA - AT^* : T \in \mathcal{B}(H)\}$$

is a real subspace of $\mathcal{B}(H)$. Thus it is quite reasonable to ask for which A will $\mathcal{R}_A$ be a complex subspace. In the case when A is normal the following theorem provides an answer to the previous question in a more general setting, $\overline{\mathcal{R}_A}$ denoting the closure of $\mathcal{R}_A$ in the operator norm topology.

Theorem 1. *Let $A \in \mathcal{B}(H)$ be a normal operator. $\overline{\mathcal{R}_A}$ is a complex subspace if and only if $A = 0$.*

Proof: Let us suppose that for the normal operator $0 \neq A \in \mathcal{B}(H)$ the subspace $\overline{\mathcal{R}_A}$ is complex. Then with $T = iI/2$ we have $-A = i(TA - AT^*) \in \overline{\mathcal{R}_A}$. Hence there is a sequence $(T_n)_n$ in $\mathcal{B}(H)$ such that

$$T_nA - AT_n^* \xrightarrow{n\to\infty} -A.$$

Clearly, we can assume that $\|-A-(T_nA - AT_n^*)\| < 1/n$ and $T_n \neq 0$ holds for every $n \in \mathbb{N}$.

By the triangular inequality it can be seen easily that for an arbitrary $B \in \mathcal{B}(H)$ the relations

$$\|\operatorname{Re} B\|, \|\operatorname{Im} B\| \leq \|B\|$$

hold, where $\operatorname{Re} B$ and $\operatorname{Im} B$ denote the real and imaginary part of B, respectively.

As it can be readily checked

$$\operatorname{Re}(-A - (T_nA - AT_n^*)) = -A_1 - i(T_nA_2 - A_2T_n^*)$$

and

$$\operatorname{Im}(-A - (T_nA - AT_n^*)) = -A_2 + i(T_nA_1 - A_1T_n^*)$$

hold where A_1 and A_2 are the real and imaginary part of A, respectively. Then the relations

$$\|A_1 + i(T_nA - AT_n^*)\|, \|A_2 - i(T_nA_1 - A_1T_n^*)\| < 1/n \qquad (n \in \mathbb{N})$$

follow from the previous remark.

Now let $\lambda \in \sigma(A)$ and moreover $\mu_1 = \operatorname{Re}\lambda$, $\nu_1 = \operatorname{Im}\lambda$. Applying the spectral theorem for normal operators there exist operators

$$B_n = \sum_{j=1}^{m_n} \mu_j^{(n)} P_j^{(n)}, \quad C_n = \sum_{j=1}^{m_n} \nu_j^{(n)} P_j^{(n)}$$

where $\mu_j^{(n)}, \nu_j^{(n)} \in \mathbb{R}$ and $P_j^{(n)}$ $(1 \leq j \leq m_n)$ are pairwise orthogonal projections for every $n \in \mathbb{N}$, such that $\mu_1^{(n)} = \mu_1$, $\nu_1^{(n)} = \nu_1$ and

$$\|A_1 - B_n\|, \quad \|A_2 - C_n\| < \min\{1/(2n\|T_n\|), 1/n\}$$

are valid. We can carry out the following estimations with respect to the operators defined above

$$\|B_n + i(T_nC_n - C_nT_n^*)\| \leq \\ \|A_1 + i(T_nA_2 - A_2T_n^*)\| + \|(B_n - A_1) + i(T_n(C_n - A_2) - (C_n - A_2)T_n^*)\| < \\ 1/n + \|A_1 - B_n\| + 2\|T_n\|\|C_n - A_2\| < 3/n$$

and

$$\|C_n - i(T_nB_n - B_nT_n^*)\| \leq \\ \|A_2 - i(T_nA_1 - A_1T_n^*)\| + \|(C_n - A_2) - i(T_n(B_n - A_1) - (B_n - A_1)T_n^*)\| < 3/n.$$

Let us suppose that $f_n \in rng\, P_1^{(n)}$ $(n \in \mathbb{N})$ are unit vectors. Since for arbitrary $\lambda \in \sigma(A)$ and $\varepsilon > 0$ the projection $E(D(\lambda, \varepsilon)) \neq 0$, where E is the spectral measure corresponding to A and $D(\lambda, \varepsilon)$ denotes the open disc in the plane with center λ and radius ε, thus the projections $P_1^{(n)}$ can be chosen in a way that $P_1^{(n)} \neq 0$ should hold for every $n \in \mathbb{N}$. In this case, with $\alpha_n = \langle T_nf_n, f_n\rangle$, the following inequalities can be established

$$(1.1) \qquad |\mu_1 + i(\alpha_n\nu_1 - \overline{\alpha}_n\nu_1)| = |\langle(B_nf_n + i(T_nC_n - C_nT_n^*)f_n, f_n\rangle| \leq \\ \|B_n + i(T_nC_n - C_nT_n^*)\| < 3/n.$$

Likewise

$$(1.2) \qquad |\nu_1 - i(\alpha_n\mu_1 - \overline{\alpha}_n\mu_1)| = |\langle(C_n - i(T_nB_n - B_nT_n^*))f_n, f_n\rangle| < 3/n.$$

Let us assume $\mu_1\nu_1 \neq 0$. Then, dividing (1.1) by ν_1 and (1.2) by μ_1 and adding together the results obtained we have

$$|\mu_1/\nu_1 + \nu_1/\mu_1| \leq \\ |\mu_1/\nu_1 + i(\alpha_n - \overline{\alpha}_n)| + |\nu_1/\mu_1 - i(\alpha_n - \overline{\alpha}_n)| < 3/n\nu_1 + 3/n\mu_1.$$

The right-hand side of the above inequality tending to zero and on the left-hand side being a positive constant, we have arrived at a contradiction. Thus $\mu_1\nu_1 = 0$. Then either μ_1 or ν_1 equals 0, which implies, considering (1.1) and (1.2), that the other one must be 0, too. Hence $\lambda = 0$. So or assumption that for the normal

A the subspace $\overline{\mathcal{R}}_A$ is a complex one yields $\sigma(A) = \{0\}$, which is equivalent to $A = 0$. $\square$

To answer the above question for an arbitrary $A \in \mathcal{B}(H)$ seems to be a more difficult task. Obviously, the condition $A = 0$ as a sufficient and necessary condition is out of the question in this case. For, if x and y are independent vectors in H, the range of the Jordan *-derivation induced by $A = x \otimes y$ is a complex subspace. It would be an interesting problem to examine the above question for a weighted unilateral shift, that is, if $U = \sum_{\gamma \in \Gamma} \alpha_\gamma e_\gamma \otimes e_{\gamma+1}$, where $(e_\gamma)_{\gamma \in \Gamma}$ is an orthonormal basis for H and $\alpha_\gamma \in \mathbb{C}$ $(\gamma \in \Gamma)$, what the necessary and sufficient conditions are for $\mathcal{R}_U$ or, furthermore, for $\overline{\mathcal{R}}_U$ to be a complex subspace.

The following theorem is concerned with a question which is close to the previous one in approach in the sense that it reveals further "skew" properties of Jordan *-derivations. Note that if $A \in \mathcal{S}(H)$ where $\mathcal{S}(H)$ stands for the set of all symmetric operators of $\mathcal{B}(H)$, then $TA - AT^*$ is skew-symmetric for every $T \in B$, that is, $\mathcal{R}_A \subset i\mathcal{S}(H)$. Similarly, if $A \in i\mathcal{S}(H)$, then $\mathcal{R}_A \subset \mathcal{S}(H)$ holds. For which A can be $\mathcal{R}_A$ the whole $\mathcal{S}(H)$ or $i\mathcal{S}(H)$, respectively?

Theorem 2. *Let H be a complex separable Hilbert space. With the above notation $\mathcal{R}_A = \mathcal{S}(H)$ if and only if A is a skew-symmetric, invertible operator. In the same way, $\mathcal{R}_A$ coincides with $i\mathcal{S}(H)$ if and only if A is an invertible, symmetric operator.*

PROOF: Let us deal with the first statement of the theorem only, as the second one can be proved analogously. Assume that A is an invertible, skew-symmetric operator. According to the above remark $\mathcal{R}_A \subset \mathcal{S}(H)$ holds. For an arbitrary $S \in \mathcal{S}(H)$

$$(SA^{-1})A - A(SA^{-1})^* = S + S = 2S,$$

thus $\mathcal{R}_A = \mathcal{S}(H)$ is indeed valid.

Conversely, let us suppose that $\mathcal{R}_A = \mathcal{S}(H)$. In this case, since

$$(iI)A - A(iI)^* = 2iA \in \mathcal{S}(H)$$

we come to the conclusion that $A \in i\mathcal{S}(H)$. Now, in order to demonstrate the invertibility of A, it is enough to prove that A is left-invertible. In accordance with the Banach theorem stating the invertibility of a linear bijection between Banach spaces, an operator in $\mathcal{B}(H)$ is left-invertible if and only if it is bounded below. Thus it is enough to show the latter property for the above operator A. Prior to this, let us prove that A is injective. Suppose that $x \neq 0 \in \mathit{ker}\, A$. We can assume that $\|x\| = 1$. On account of the relation $\mathcal{R}_A = \mathcal{S}(H)$ there is a $T \in \mathcal{B}(H)$ such that

$$TA - AT^* = x \otimes x.$$

At the same time

$$\langle (x \otimes x)x, x\rangle - \langle (TA - AT^*)x, x\rangle = 1 - (\langle TAx, x\rangle + \langle T^*x, Ax\rangle) = 1,$$

which is a contradiction.

Let us suppose now that A is not bounded below. Then there is a sequence $(e_n)_n$ in H such that $\|e_n\| = 1$ ($n \in \mathbb{N}$) and

$$Ae_n \xrightarrow{n\to\infty} 0.$$

Applying the well-known fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence, we can assume that $e_n \xrightarrow{w} e$ for some $e \in H$. Choosing an arbitrary $y \in H$

$$\langle Ae_n, y\rangle = \langle e_n, A^*y\rangle \xrightarrow{n\to\infty} \langle e, A^*y\rangle.$$

On the other hand, since $Ae_n \longrightarrow 0$ implies $\langle Ae_n, y\rangle \longrightarrow 0$,

$$\langle e, A^*y\rangle = 0 \qquad (y \in H). \tag{2.1}$$

Owing to the relation $\overline{rng\, A^*} = ker\, A^{\perp} = \{0\}^{\perp} = H$ the operator A^* has dense range and thus (2.1) can be valid only if $e = 0$.

We shall make use of F. Wolf's theorem (cf. [4]), which claims that if $e_n \xrightarrow{w} 0$ and $Ae_n \longrightarrow 0$, then the sequence $(e_n)_n$ can be chosen to be orthonormal. In other words, with the notation above, for the A in question there is an orthonormal sequence $(e_n)_n$ for which $Ae_n \longrightarrow 0$. Let $P = \sum_{n=1}^{\infty} e_n \otimes e_n$. Then for every $T \in \mathcal{B}(H)$

$$\langle Pe_n, e_n\rangle - \langle (TA - AT^*)e_n, e_n\rangle =$$
$$\langle Pe_n, e_n\rangle - \langle TAe_n, e_n\rangle - \langle T^*e_n, Ae_n\rangle \xrightarrow{n\to\infty} 1,$$

hence $P \notin \mathcal{R}_A$. Thus A must be bounded below, the proof is completed. □

We end up the examination of the present question with an open problem. Namely, what can be said about A if $\mathcal{S}(H) \subsetneq \mathcal{R}_A$, that is, $\mathcal{S}(H)$ is a proper subset of $\mathcal{R}_A$?

In the last part of the paper we shall treat a question in regard to generalized Jordan *-derivations. The antecedent of the notion of generalized Jordan *-derivations was the concept of Jordan *-derivation pairs, which was introduced by Zalar [19] giving a more natural proof than Šemrl with respect to the problem of representing quadratic forms by sesquilinear ones on modules over *-rings. Molnár has shown ([8]) that on standard operator algebras Jordan *-derivation pairs are of the form

$$E(T) = TA - BT^*, \qquad F(T) = TB - AT^* \qquad (T \in \mathcal{B}(H))$$

for some operators $A, B \in \mathcal{B}(H)$.

For the case of generalized derivations Fialkow and Loebl [3] proved that the range

$$\mathcal{J}_{AB} = \{TA - BT : T \in \mathcal{B}(H)\}$$

of the generalized derivation $T \mapsto TA - BT$ is contained in an ideal $\mathcal{I}$ of $\mathcal{B}(H)$ if and only if both A and B are in $\mathcal{I}$. In the present situation, if H is a complex Hilbert space, one can easily check that exactly the same conditions work for a similar question pertinent to generalized Jordan *-derivations. The problem is getting more complicated if we demand equality instead of inclusion, in other words: what can be said about A and B if $\mathcal{R}_{AB} = \mathcal{I}$, where $\mathcal{I}$ is an ideal in $\mathcal{B}(H)$.

Theorem 3. *Let H be an infinite dimensional, complex, separable Hilbert space and $A, B \in \mathcal{B}(H)$. Then $\mathcal{R}_{AB}$ cannot coincide with a non-zero proper ideal of $\mathcal{B}(H)$.*

Proof: Let us suppose that for $A, B \in \mathcal{B}(H)$ the range $\mathcal{R}_{AB}$ is a non-zero proper ideal of $\mathcal{B}(H)$. Then by a well-known theorem of Calkin we have $\mathcal{F}(H) \subset \mathcal{R}_{AB} \subset \mathcal{C}(H)$. Because of the ideal property of $\mathcal{R}_{AB}$ we also have

$$i(TA - BT^*) + (iT)A - B(iT)^* = 2iTA \in \mathcal{R}_{AB}$$

for every $T \in \mathcal{B}(H)$, thus $A \in \mathcal{R}_{AB}$. It can be proved in a similar way that $B \in \mathcal{R}_{AB}$ holds, too. Let $0 \neq K \in \mathcal{R}_{AB}$ be arbitrary. Since H is infinite dimensional and K is compact, $K \neq \lambda I$ for any $\lambda \in \mathbb{C}$. In this case, according to a result of Radjavi and Rosenthal [11, Theorem 2] there is an orthonormal basis $\{e_n\}_n$ in H for which

$$\langle Ke_n, e_m \rangle \neq 0 \qquad (n, m \in \mathbb{N}).$$

As A and B^* are compact operators, they map weakly convergent sequences into norm convergent ones, hence $\|Ae_n\|, \|B^*e_n\| \longrightarrow 0$. Let $\alpha_n = \langle Ke_n, e_n \rangle \neq 0$. Now, as in the proof of [16, Theorem 2], let $(e_{k_n})_n$ be a subsequence of $(e_n)_n$ such that

$$\|Ae_{k_n}\| \leq |\alpha_n|/n \qquad (n \in \mathbb{N})$$

and

$$\|B^*e_{k_n}\| \leq |\alpha_n|/n \qquad (n \in \mathbb{N})$$

hold. Let U be a partial isometry defined as follows

$$Ue_{k_n} = e_n \qquad (n \in \mathbb{N})$$

and U is zero on the orthogonal complement of the subspace generated by the vectors $\{e_{k_n}\}_n$. Since $\mathcal{R}_{AB}$ is an ideal there is a $T \in \mathcal{B}(H)$ for which

$$TA - BT^* = U^*KU \in \mathcal{R}_{AB}.$$

With this T we obtain

$$|\alpha_n| = |\langle Ke_n, e_n\rangle| = |\langle U^*KUe_{k_n}, e_{k_n}\rangle| \leq$$
$$|\langle TAe_{k_n}, e_{k_n}\rangle| + |\langle T^*e_{k_n}, B^*e_{k_n}\rangle| \leq 2\|T\||\alpha_n|/n.$$

It follows from the inequalities above that $\|T\| \geq n/2$ for every $n \in \mathbb{N}$, which contradicts the boundedness of T. $\square$

Remark. We would mention that $\mathcal{R}_{AB} = \mathcal{B}(H)$ may occur. This is the case, for example, if A is an invertible operator and $B = 0$ or vice versa. We cannot, however, give a necessary and sufficient condition for the surjectivity of $\mathcal{R}_{AB}$.

References

[1] Apostol C., Stampfli J.G., *On derivation ranges*, Indiana Univ. Math. J. **25** (1976), 857–869.
[2] Brešar M., Zalar B., *On the structure of Jordan *-derivations*, Colloquium Math. **63** (1992), 163–171.
[3] Fialkow L.A., Loebl R., *Elementary mappings into ideals of operators*, Ill. J. Math. **28** (1984), 555–578.
[4] Fillmore P.A., Stampfli J.G., Williams J.P., *On the essential numerical range, the essential spectrum, and a problem of Halmos*, Acta Sci. Math. **33** (1972), 179–192.
[5] Johnson B.E., Williams J.P., *The range of a normal derivation*, Pacific J. Math. **58** (1975), 105–122.
[6] Molnár L., *The range of a Jordan *-derivation*, preprint.
[7] Molnár L., *On the range of a normal Jordan *-derivation*, Comment. Math. Univ. Carolinae **35** (1994), 691–695.
[8] Molnár L., *Jordan *-derivation pairs on a complex *-algebra*, preprint.
[9] Molnár L., *A condition for a subspace of $\mathcal{B}(H)$ to be an ideal*, Linear Algebra and Appl., to appear.
[10] Molnár L., *The range of a Jordan *-derivation on an H^*-algebra*, preprint.
[11] Radjavi H., Rosenthal P., *Matrices for operators and generators of $\mathcal{B}(H)$*, J. London Math. Soc. **2** (1970), 557–560.
[12] Šemrl P., *On Jordan *-derivations and an application*, Colloquium Math. **59** (1990), 241–251.
[13] Šemrl P., *Quadratic functionals and Jordan *-derivations*, Studia Math. **97** (1991), 157–165.
[14] Šemrl P., *Quadratic and quasi-quadratic functionals*, Proc. Amer. Math. Soc. **119** (1993), 1105–1113.
[15] Šemrl P., *Jordan *-derivations of standard operator algebras*, Proc. Amer. Math. Soc. **120** (1994), 515–518.
[16] Stampfli J.G., *Derivations on $\mathcal{B}(\mathcal{H})$: The range*, Ill. J. Math. **17** (1973), 518–524.
[17] Stampfli J.G., *On the range of a hyponormal derivation*, Proc. Amer. Math. Soc. **52** (1975), 117–120.
[18] Williams J.P., *Derivations ranges: open questions*, Topics in Modern Operator Theory (Timisoara/ Herculane, 1980), Birkhäuser, Basel-Boston, Mass., 1981, pp. 319–328.
[19] Zalar B., *Jordan *-derivation pairs and quadratic functionals on modules over *-rings*, preprint.

Szegfű u. 5. III/10, H-4027 Debrecen, Hungary

E-mail: battya@tigris.klte.hu

(*Received* May 30, 1995)