

## The link between the kernel method and the method of adjoints for the generalized index ${}_2F_1$ -transform

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*Abstract.* In this note we show that the two definitions of generalized index  ${}_2F_1$ -transform given in the previous works [1] and [2] agree for distributions of compact support.

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The generalized index  ${}_2F_1$ -transform of  $f \in \mathcal{E}'(I)$ ,  $I = (0, \infty)$ , was defined by the kernel method in [1] as:

$$(1) \quad F(\tau) = \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle, \quad \tau \in I,$$

where

$$\mathbf{F}(\mu, \alpha, \tau, x) = {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -x\right)x^\alpha$$

and  ${}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -x)$  is the Gauss hypergeometric function defined for  $|x| < 1$  as the sum of the Gauss series, while for  $|x| \geq 1$ , is defined as its analytic continuation [4, p. 431]. Here  $\mu$  and  $\alpha$  are complex parameters.

In [2] the generalized index  ${}_2F_1$ -transform was analyzed by the method of adjoints. It was required to introduce two Fréchet spaces  $V$  and  $W$  satisfying that the operator  $\mathcal{L} : W \rightarrow V$  given by

$$(2) \quad (\mathcal{L}\psi)(x) = \int_0^\infty \psi(\tau)\mathbf{F}(\mu, \alpha, \tau, x) d\tau, \quad x > 0,$$

was continuous whenever  $Re \mu > -\frac{1}{2}$ . To be more precise, the spaces  $V$  and  $W$  were defined by

$$V = \{ \phi \in \mathcal{C}^\infty(I) : \gamma_k(\phi) < \infty, \quad k \in \mathbb{N} \cup \{0\} \}$$

where

$$\gamma_k(\phi) = \sup_{x \in I} \left| x^{-\alpha}(x+1)^{\mu+\frac{3}{2}} A_x^k \phi(x) \right|,$$

$A_x$  being the differential operator defined by

$$x^{\alpha-\mu}(x+1)^{-\mu}D_x x^{\mu+1}(x+1)^{\mu+1}D_x x^{-\alpha},$$

and

$$W = \{\psi \in \mathcal{C}^\infty(\mathbb{R}), \text{ even} : \varrho_r(\psi) < \infty, r \in \mathbb{N} \cup \{0\}\},$$

where

$$\varrho_r(\psi) = \sup_{x \in I} \left| x^{\frac{3}{2}} e^x \left( \mathcal{F} \left( \frac{[(\mu + \frac{1}{2})^2 + \tau^2]^r \psi(\tau)}{\tau \Gamma(\mu + \frac{1}{2} + i\tau) \Gamma(\mu + \frac{1}{2} - i\tau)} \right) \right) (x) \right|,$$

and  $\mathcal{F}$  denotes Fourier transform with respect to the  $\tau$ -variable.

The generalized index  ${}_2F_1$ -transform was defined as the adjoint operator of  $\mathcal{L}$ , namely,

$$(3) \quad \langle \mathcal{L}' f, \psi \rangle = \langle f, \mathcal{L} \psi \rangle, \quad f \in V', \quad \psi \in W.$$

In this note we establish the link between these two definitions. Here, it will be proved that both definitions agree for distributions of compact support. The corresponding result for the Kontorovich-Lebedev transform was established in [3, Proposition 2.4].

**Theorem.** *Let  $f \in \mathcal{E}'(I)$  and  $\alpha, \mu$  being complex parameters with  $Re \alpha > 0$ ,  $Re \mu > 0$ ,  $\frac{1}{8} < Re(\mu - \alpha) < \frac{1}{4}$  and  $Re(\mu - 2\alpha) < -1$ . Then, for any  $\psi \in W$ , one has*

$$(4) \quad \langle \mathcal{L}' f, \psi \rangle = \langle T_{\langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle}, \psi(\tau) \rangle,$$

where  $T_{\langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle}$  is the member of  $W'$  given by

$$(5) \quad \langle T_{\langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle}, \psi(\tau) \rangle = \int_0^\infty \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle \psi(\tau) d\tau.$$

PROOF: First, observe that  $\mathcal{D}(I) \subset V \subset \mathcal{E}(I)$  and the topology of  $V$  is stronger than the one induced on it by  $\mathcal{E}(I)$ . Thus,  $\mathcal{E}'(I)$  is a subspace of  $V'$ .

Also, from [1, Theorem 3.2, p. 662] the function  $F(\tau)$  given by (1) is an entire function such that

- (i)  $F(\tau) = O(1)$  as  $\tau \rightarrow 0^+$ ;
- (ii) there exists a  $r \in \mathbb{N} \cup \{0\}$  such that  $F(\tau) = O\left(\tau^{2r - Re \mu - \frac{1}{2}}\right)$  as  $\tau \rightarrow \infty$ .

From this and having into account that, for any  $\psi \in W$  one has

- (i)  $\psi(\tau) = O(\tau^2)$  as  $\tau \rightarrow 0^+$ ;
- (ii) for all  $p \in \mathbb{N}$ ,  $\psi(\tau) = O(\tau^{-p})$ , as  $\tau \rightarrow \infty$ ;

(cf. [2, Proposition 2.2 and Proposition 2.4]) it follows that the integral of the right hand side of (5) has sense.

Clearly, if  $f \in \mathcal{D}(I)$ , the result of this Theorem holds. In fact, assume that  $f$  has its support contained in the closed interval  $[a, b] \subset I$ , then, using Fubini theorem, one has

$$\begin{aligned} \langle \mathcal{L}' f, \psi \rangle &= \langle f, \mathcal{L} \psi \rangle \\ &= \int_a^b f(x) (\mathcal{L} \psi)(x) dx = \int_a^b f(x) \int_0^\infty \mathbf{F}(\mu, \alpha, \tau, x) \psi(\tau) d\tau dx \\ &= \int_0^\infty \psi(\tau) \int_a^b f(x) \mathbf{F}(\mu, \alpha, \tau, x) dx d\tau = \int_0^\infty F(\tau) \psi(\tau) d\tau. \end{aligned}$$

So, the result holds for  $f \in \mathcal{D}(I)$ .

Now, for  $f \in \mathcal{E}'(I)$  and using [5, Theorem 28.2(i), p. 301], there exists a sequence of functions  $\{f_n\}_{n \in \mathbb{N}}$  belonging to  $\mathcal{D}(I)$  which converges to  $f$  in  $\mathcal{E}'(I)$  and so, in  $V'$ . Furthermore, from [2, Proposition 2.3] it follows that  $\mathcal{L}'$  is a continuous mapping from  $V'$  into  $W'$ . Thus  $\{\mathcal{L}' f_n\}_{n \in \mathbb{N}}$  converges to  $\mathcal{L}' f$  in  $W'$  as  $n \rightarrow \infty$ .

Since, for all  $\psi \in W$ ,

$$\langle \mathcal{L}' f_n, \psi \rangle = \int_0^\infty F_n(\tau) \psi(\tau) d\tau$$

where

$$F_n(\tau) = \langle f_n(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle,$$

it follows that

$$(6) \quad \langle \mathcal{L}' f, \psi \rangle = \lim_{n \rightarrow \infty} \int_0^\infty F_n(\tau) \psi(\tau) d\tau.$$

Clearly, from the asymptotic growth of  $F_n$  and  $\psi$ , the limit in (6) goes into the integral. Also, noting that

$$\begin{aligned} \lim_{n \rightarrow \infty} F_n(\tau) &= \lim_{n \rightarrow \infty} \langle f_n(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle \\ &= \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle = F(\tau), \end{aligned}$$

the result holds. □

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