Dimension and ε-translations

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

Abstract. Some theorems characterizing the metric and covering dimension of arbitrary subspaces in a Euclidean space will be obtained in terms of ε-translations; some of them were proved in our previous paper [G1] under the additional assumption of the boundedness of subspaces.

Keywords: metric dimension, covering dimension, ε-translation, uniformly 0-dimensional mappings

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1. Introduction

In the previous paper [G1] we proved some theorems which characterize the metric dimension μdim for bounded subspaces in a Euclidean space in terms of ε-translations. In this paper, these results will be extended for arbitrary (unbounded) subspaces and also, we will obtain some results characterizing the covering dimension dim in terms of some classes of ε-translations such as $\mathcal{U}$ -0-dimensional mappings in the sense of [Z-S] or uniformly 0-dimensional mappings of Katětov [Ka1].

Throughout this paper, all spaces are assumed to be *metric* and mappings are *continuous.*

2. Metric dimension and ε-translations

Let $X \subseteq \mathsf{R}^n$ and $\varepsilon > 0$. Then a mapping $f : X \to \mathsf{R}^n$ is called an *ε-translation* if $\| x - f(x) \| < \varepsilon$ for every $x \in X$. The *metric dimension* μdim X of X is defined to be the least integer m for which X admits open covers of order $\leq m+1$ with arbitrarily small meshes [Sm1]. Suppose $\mathcal{U}$ is a locally finite open cover of X and $\mathcal{P} = \{p_U : U \in \mathcal{U}\}$ is an arbitrary set in R^n. Consider the rectilinear closed (degenerate in general) simplex $(p_{U_0}, \dots, p_{U_r})$ with vertices $p_{U_0}, \dots, p_{U_r}$ for every finite number of elements $U_0, \dots, U_r \in \mathcal{U}$ with $U_0 \cap \dots \cap U_r \neq \emptyset$. Let $\mathcal{N}$ be the family of all of these simplexes and we call $\mathcal{N}$ the *complex determined by $\mathcal{U}$ and $\mathcal{P}$* . Then the *κ-mapping* $f : X \to \cup\mathcal{N}$ relative to $\mathcal{U}$ and $\mathcal{P}$ is defined by

$$f(x) = \sum_{U \in \mathcal{U}} f_U(x) p_U \quad \text{where} \quad f_U(x) = \frac{d(x, X-U)}{\sum_{V \in \mathcal{U}} d(x, X-V)} \quad \text{for} \quad x \in X.$$

If for some $\varepsilon > 0$, $\delta(U \cup \{p_U\}) < \varepsilon$ for every $U \in \mathcal{U}$, then f is an ε-translation. By a *simplicial complex* $\mathcal{K}$ in R^n, we mean a geometric (not necessarily finite) simplicial complex which is locally finite in R^n at every point in $\cup\mathcal{K}$. Also, a *polyhedron* means an underlying space of a simplicial complex. If $P = \cup\mathcal{K}$ and $\mathcal{K}$ is a uniform complex in the sense of Smirnov [Sm2], then we call $\mathcal{K}$ a *uniform triangulation* of P. We note that if a polyhedron P in R^n admits a uniform triangulation, then P is closed in R^n [Sm2]. The following lemma is an extension of [Eg, Theorem 3].

Lemma 1. *Let X be an arbitrary subspace in R^n with $\mu\dim X \leq m$, $0 \leq m \leq n-1$. Then for every $\varepsilon > 0$ and every sequence $\{H_i\}$ of $(n-m-1)$-dimensional planes in R^n, there exists an ε-translation of $f : X \to P \subseteq \mathsf{R}^n - \cup H_i$ where P is an m-dimensional polyhedron with a uniform triangulation.*

Proof: Take a $\delta > 0$ with $4\sqrt{n}\delta/3 < \varepsilon$. For every integer k, we denote by $E(k)$ the open interval $((k-\frac{2}{3})\delta, (k+\frac{2}{3})\delta)$ and set

$$\mathcal{E} = \{E(k_1,\dots,k_n) : k_1,\dots,k_n \in \mathsf{Z}\} \text{ where } E(k_1,\dots,k_n) = E(k_1) \times \cdots E(k_n).$$

Then $\mathcal{E}$ is an open cover of R^n by open n-cubes with mesh $< \varepsilon$. For $E \in \mathcal{E}$, we denote by p_E the center of E and set $\mathcal{P} = \{p_E : E \in \mathcal{E}\}$. Let $\mathcal{N}$ be the complex determined by $\mathcal{E}$ and $\mathcal{P}$. Denote by $\tau(k_1,\dots,k_n)$ the closed n-cube $\{x \in \mathsf{R}^n : k_i\delta \leq x_i \leq (k_i+1)\delta\ , 1 \leq i \leq n\}$, and we set $\mathcal{T} = \{\tau(k_1,\dots,k_n) : k_1,\dots,k_n \in \mathsf{Z}\}$. Then for every simplex $\sigma \in \mathcal{N}$ there exists $\tau \in \mathcal{T}$ such that all vertices of σ are those of τ. For $\tau \in \mathcal{T}$, let V_τ be the set of vertices of τ. Then the family of all $(n-1)$-dimensional planes determined by n points from V_τ defines a cellular decomposition of τ, and applying the barycentric decomposition [AH], we obtain a simplicial decomposition $\mathcal{K}_\tau$ of τ. Then $\mathcal{K} = \cup\{\mathcal{K}_\tau : \tau \in \mathcal{T}\}$ defines a uniform triangulation of R^n since every $\mathcal{K}_\tau$ is finite and congruent to each other.

Now let $\mathcal{U}$ be an open cover of X with mesh $\mathcal{U} < \delta/3$ and ord $\mathcal{U} \leq m+1$; such a cover exists since $\mu\dim X \leq m$ by assumption. Since $\delta/3$ is a Lebesgue number of $\mathcal{E}$ there exists $i : \mathcal{U} \to \mathcal{E}$ such that $U \subseteq i(U)$ for every $U \in \mathcal{U}$. Define an open cover $\mathcal{V}$ of X by

$$\mathcal{V} = \{V_E : E \in \mathcal{E}\} \quad \text{where} \quad V_E = \cup\{U \in \mathcal{U} : i(U) = E\}.$$

Then $\mathcal{V}$ is a star-finite open cover of X and ord $\mathcal{V} \leq m+1$. Let $\mathcal{L}$ be the complex determined by $\mathcal{V}$ and $\mathcal{P}$ and $g : X \to \cup\mathcal{L}$ the κ-mapping relative to $\mathcal{V}$ and $\mathcal{P}$. Note that $\cup\mathcal{L} \subseteq \cup\mathcal{K}^{(m)}$ where $\mathcal{K}^{(m)}$ is the m-skeleton of $\mathcal{K}$ and that g is a λ-translation where $\lambda =$ mesh $\mathcal{E}$, because $\delta(V_E \cup \{p_E\}) \leq \delta(E)$ for $E \in \mathcal{E}$. Let $V_0 = \{p_i\}$ be the set of vertices in $\mathcal{K}^{(m)}$. Since $\mathcal{K}$ is uniform, so is $\mathcal{K}^{(m)}$. Hence by [Sm2, Corollary to Theorem 2] there exists an $\varepsilon' > 0$ satisfying the condition:

> if $\{q_i\} \subseteq \mathsf{R}^n$ and $\| p_i - q_i \| < \varepsilon'$ for every i, then there exist a uniform complex $\mathcal{K}'$ with vertices in $\{q_i\}$ and an isomorphism $\varphi : \mathcal{K}^{(m)} \to \mathcal{K}'$ sending each simplex $(p_{i_0},\dots,p_{i_r})$ to $(q_{i_0},\dots,q_{i_r})$.

We may assume that $\lambda + \varepsilon' < \varepsilon$. Moreover, by [Ku, p. 307] we can choose $\{q_i\}$ so that $\{q_i\}$ is in general position relative to $\{H_i\}$ i.e., $\sigma \cap (\cup H_i) = \emptyset$ for every simplex σ whose vertices are in $\{q_i\}$ and $\dim \sigma \leq m$. Then the polyhedron $P = \cup \mathcal{K}'$ is disjoint from $\cup H_i$ and the homeomorphism $h : \cup \mathcal{K}^{(m)} \to P$ induced from φ, which is linear on each simplex, is an ε'-translation. Then $f = h \circ g : X \to P$ is a desired ε-translation since $\| x - f(x) \| \leq \| x - g(x) \| + \| g(x) - h(g(x)) \| < \lambda + \varepsilon' < \varepsilon$ for every $x \in X$. □

Let m, n be integers with $0 \leq m \leq n-1$. The space N^n_m is defined to be the set of points in R^n at most m of whose coordinates are rationals. Then we have $\dim \mathrm{N}^n_m = \mu\mathrm{dim}\ \mathrm{N}^n_m = m$[E]. The space S^n_m, which was defined in [G2] by modifying the space $S_{n,m}$ in [G1], satisfies the relations:

$$\mathrm{N}^n_m \subseteq \mathrm{S}^n_m,\ \mu\mathrm{dim}\ \mathrm{S}^n_m = m \text{ and } \dim \mathrm{S}^n_m = \min\{2m, n-1\}.$$

Note that $\dim X \leq 2\mu\mathrm{dim}\ X$ for every X by [Ka2]. Hence, among those subspaces in R^n of metric dimension m, S^n_m is of the maximal difference with its covering dimension.

The following theorem is an extension of [G1, Theorem 1] which was proved under the additional condition of the boundedness of X.

Theorem 2. *Let X be an arbitrary subspace in R^n and m an integer with $0 \leq m \leq n-1$. Then the following conditions are equivalent.*

(a) $\mu\mathrm{dim}\ X \leq m$.

(b) *For every $\varepsilon > 0$ and every polyhedron P in R^n of dimension $\leq n-m-1$, there exists an ε-translation $f : X \to \mathsf{R}^n$ with $f(X) \cap P$ (or $\mathrm{Cl}(f(X)) \cap P) = \emptyset$.*

(c) *For every $\varepsilon > 0$ and every polyhedron P with a uniform triangulation in R^n of dimension $\leq n-m-1$, there exists an ε-translation $f : X \to \mathsf{R}^n$ with $f(X) \cap P$ (or $\mathrm{Cl}(f(X)) \cap P) = \emptyset$.*

Proof: Since every polyhedron admits a triangulation consisting of countably many simplexes, (a) implies (b) by Lemma 1. Obviously (b) implies (c).

Assume that the condition (c) is satisfied. Then for every $\varepsilon > 0$, as was proved essentially in [G1, Theorem 1], there exists an ε-translation of X into an m-dimensional polyhedron; it needs only to observe that the polyhedron $B_{i,n-m-1}$ in [G1] allows a uniform triangulation. Hence by [Sm1, Corollary 2] we have $\mu\mathrm{dim}\ X \leq m$. □

The following theorem which extends [G1, Theorem 2], can be proved similarly by use of Lemma 1 and its proof is omitted.

Theorem 3. *For every subspace X in R^n and every integer m with $0 \leq m \leq n-1$, the following conditions are equivalent.*

(a) $\mu\mathrm{dim}\ X \leq m$.

(b) *For every $\varepsilon > 0$ there exists an ε-translation f of X into an m-dimensional polyhedron P (with a uniform triangulation) such that $P \subseteq \mathrm{N}^n_m$.*

(c) *For every* $\varepsilon > 0$ *there exists an* ε*-translation* f *of* X *into an* m*-dimensional polyhedron* P (*with a uniform triangulation*) *such that* $P \subseteq \mathrm{S}^n_m$.

3. Covering dimension and ε-translations

Let $\mathcal{U}$ be an open cover of a space X and $A \subseteq X$. Then we write $\mathcal{U}$-$\dim A \leq 0$ if there exists a pairwise disjoint open collection $\mathcal{U}_0$ in X such that $\cup\mathcal{U}_0 \supseteq A$ and $\mathcal{U}_0$ refines $\mathcal{U}$. A mapping $f : X \to Y$ is called $\mathcal{U}$*-0-dimensional* (or $\mathcal{U}$-$\dim f \leq 0$) if for some open cover $\mathcal{V}$ of Y, $\mathcal{U}$-$\dim f^{-1}(V) \leq 0$ for every $V \in \mathcal{V}$ [Z-S].

Lemma 4. *Let* $\mathcal{U}$ *be a countable star-finite open cover of a space* X *with* $\operatorname{ord}\mathcal{U} \leq k+1$ *and* $\mathcal{N}$ *the complex determined by* $\mathcal{U}$ *and* $\mathcal{P} = \{p_U : U \in \mathcal{U}\} \subseteq \mathsf{R}^n$. *If* $\mathcal{N}$ *consists of non-degenerate simplexes and is locally finite in* R^n *at every point in* $\cup\mathcal{N}$, *then* $\mathcal{U}$-$\dim f \leq 0$ *for the* κ*-mapping* f *determined by* $\mathcal{U}$ *and* $\mathcal{P}$.

Proof: By [Ku, p. 239], there exists a geometric realization $\mathcal{K}$ of the nerve of $\mathcal{U}$ in R^{2k+1}. Let $\mathcal{Q} = \{q_U : U \in \mathcal{U}\}$ where q_U is the vertex of $\mathcal{K}$ corresponding to $U \in \mathcal{U}$, and let $\pi : \mathcal{K} \to \mathcal{N}$ be the mapping sending each simplex $(q_{U_0}, \dots, q_{U_r})$ to $(p_{U_0}, \dots, p_{U_r})$. Since $\mathcal{K}$ is locally finite, π induces a mapping $p : \cup\mathcal{K} \to \cup\mathcal{N}$ uniquely which is linear on each simplex in $\mathcal{K}$. Clearly we have $f = p \circ g$ for the κ-mapping g relative to $\mathcal{U}$ and $\mathcal{Q}$.

Let $y \in \cup\mathcal{N}$. Then y is contained in the interior of only finitely many simplexes in $\mathcal{N}$, say $\sigma_1, \dots, \sigma_s$. Since p is homeomorphic on each simplex, $p^{-1}(y)$ consists of exactly s points. For every $z_i \in p^{-1}(y)$, we choose a simplex $\tau_i \in \mathcal{K}$ such that z_i is in the interior of τ_i, $1 \leq i \leq s$. Let W_i be the open star of τ_i in $\mathcal{K}$, and then $\{W_i : 1 \leq i \leq s\}$ is pairwise disjoint. For, if otherwise, there would be a simplex $\tau \in \mathcal{K}$ with distinct faces τ_i and τ_j. But this contradicts that p is homeomorphic on τ. Let $\mathcal{L}$ be the subcomplex of $\mathcal{K}$ such that $\cup\mathcal{L} = \cup\mathcal{K} - \cup\{W_i : 1 \leq i \leq s\}$. Since $\mathcal{N}$ is locally finite by assumption, $V_y = \cup\mathcal{N} - p(\cup\mathcal{L})$ is an open neighborhood of y such that $f^{-1}(V_y) = g^{-1}p^{-1}(V_y) \subseteq \cup\{g^{-1}(W_i) : 1 \leq i \leq s\}$. Since g is a κ-mapping, $\{g^{-1}(W_i) : 1 \leq i \leq s\}$ refines $\mathcal{U}$. This means $\mathcal{U}$-$\dim f^{-1}(V_y) \leq 0$ and hence $\mathcal{U}$-$\dim f \leq 0$. □

Theorem 5. *Let* X *be an arbitrary subspace in* R^n *and* k *an integer with* $0 \leq k \leq n$. *Then* $\dim X \leq k$ *iff for every finite open cover* $\mathcal{U}$ *of* X, *there exists an* ε*-translation* $f : X \to \mathsf{R}^n$ *such that* $\mathcal{U}$-$\dim f \leq 0$ *and* $f(X)$ (*or* $\mathrm{Cl}(f(X))$) $\subseteq \mathrm{N}^n_k$.

Proof: *Necessity.* Let $\varepsilon > 0$ and $\mathcal{U} = \{U_1, \dots, U_r\}$ be an open cover of X. Let $\mathcal{E}$ be the cover of R^n by open n-cubes with mesh $< \varepsilon$ in the proof of Lemma 1. Since $\dim X \leq k$, there exists an open cover $\mathcal{V} = \{V(k_1, \dots, k_n; j) : k_1, \dots, k_n \in \mathsf{Z}, 1 \leq j \leq r\}$ such that $\operatorname{ord}\mathcal{V} \leq k+1$ and $V(k_1, \dots, k_n; j) \subseteq E(k_1, \dots, k_n) \cap U_j$ for every k_i and j. As in the proof of Lemma 1, we can take $\mathcal{P} = \{p_V : V \in \mathcal{V}\}$ in R^n such that

$\mathcal{P}$ is in general position in R^n,
$p_V \in E(k_1, \dots, k_n)$ for $V = V(k_1, \dots, k_n; j)$, and
$\cup\mathcal{N} \subseteq \mathrm{N}^n_k$ where $\mathcal{N}$ is the complex determined by $\mathcal{V}$ and $\mathcal{P}$.

Then $\mathcal{N}$ consists of non-degenerate simplexes and is locally finite in R^n. Hence the κ-mapping f relative to $\mathcal{V}$ and $\mathcal{P}$ is $\mathcal{U}$-0-dimensional by Lemma 4 and is a desired ε-translation since $\delta(V \cup \{p_V\}) < \text{mesh } \mathcal{E} < \varepsilon$. The proof of the sufficiency is almost evident. $\square$

Let $X \subseteq \mathsf{R}^n$ and $\varepsilon > 0$. We denote by $\mathrm{T}_\varepsilon(X)$ the collection of all ε-translations of X into R^n and set $T(X) = \cup\{\mathrm{T}_\varepsilon(X) : \varepsilon > 0\}$. Then $T(X)$ is complete relative to the metric defined by $d(f,g) = \sup\{\| f(x) - g(x) \| : x \in X\}$.

Theorem 6. *Let X be a bounded subspace in R^n with $0 \le k \le n$. Then $\dim X \le k$ iff for every $\varepsilon > 0$ there exists a uniformly 0-dimensional ε-translation $f : X \to \mathsf{R}^n$ such that $f(X)$ (or $\mathrm{Cl}(f(X))) \subseteq \mathrm{N}^n_k$.*

Proof: The sufficiency of the theorem follows from the fact that every uniformly 0-dimensional mapping does not decrease the dimension [Ka1, Theorem 3.3].

Assume that $\dim X \le k$ and $\varepsilon > 0$. Let $\{H_i\}$ be a sequence of $(n-k-1)$-dimensional planes in R^n such that $\mathsf{R}^n - \mathrm{N}^n_k = \cup H_i$. We set

$\mathcal{S}_i = \{f \in \mathrm{T}(X) : \mathrm{Cl}(f(X)) \cap H_i = \emptyset\}$ for $i \in \mathsf{N}$, and

$\mathcal{T} = \{f \in \mathrm{T}(X) : f$ is uniformly 0-dimensional $\}$.

Then $\mathcal{S}_i$ is dense and open in $\mathrm{T}(X)$, and $\mathcal{T}$ is a dense G_δ-set in $\mathrm{T}(X)$ [Ka1, Theorem 2.15]. Hence $\cap\mathcal{S}_i \cap \mathcal{T}$ is dense in $\mathrm{T}(X)$, and there exists $f \in \cap\mathcal{S}_i \cap \mathcal{T}$ with $d(1_X, f) < \varepsilon$. Then f is an ε-translation of X with $\mathrm{Cl}(f(X)) \subseteq \mathrm{N}^n_k$. $\square$

We don't know whether Theorem 6 is valid for unbounded subspace X.

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