A note on topology of Z-continuous posets

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Abstract. Z-continuous posets are common generalizations of continuous posets, completely distributive lattices, and unique factorization posets. Though the algebraic properties of Z-continuous posets had been studied by several authors, the topological properties are rather unknown. In this short note an intrinsic topology on a Z-continuous poset is defined and its properties are explored.

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Introduction

Z-continuous posets were introduced by Wright, Wagner, and Thatcher [WWT] as a generalization of continuous lattices. The family of Z-continuous posets in fact includes completely distributive lattices ([R]), and unique factorization posets ([M]). The algebraic properties of Z-continuous posets had been studied by several authors eg. [BE], [N], [V1], [V2]. Though topological methods play an important role in the theory of continuous lattices from its inception, the topological properties of Z-continuous posets have never been studied. In this short note, we define an intrinsic topology on a Z-continuous poset, and point out some pleasant properties of this topology. Of course a lot more need to be done in this direction.

A *subset system* Z is a function which assigns to each poset P a set $Z(P)$ of subsets of P such that (i) for all P, all singletons of P are in $Z(P)$, and (ii) if $f : P \to Q$ is a monotone function between posets, and S is $Z(P)$, then $f(S)$ is in $Z(Q)$ ([WWT]). Some examples of the subset systems are all subsets, directed subsets, and finite subsets; see [V1] and [V2] for more examples. For $S \in Z(P)$, $\downarrow S$ is called a Z-ideal. The poset (ordered by inclusion) of all Z-ideals of a poset P is denoted by $I_Z(P)$. Let P be a poset. For $x, y \in P$, x is said to be Z-*waybelow* y (written $x \ll y$) if whenever $y \leq \sup S$ for some $S \in Z(P)$, there exists an $s \in S$ such that $x \leq s$. A poset is called Z-*continuous* if (i) it is Z-*complete* (meaning: for every $S \in Z(P)$, $\sup S$ exists), (ii) for every $x \in P$, the set $\Downarrow x = \{y : y \ll x\} \in I_Z(P)$, and for every $x \in P$, $x = \mathrm{Sup} \Downarrow x$. A Z-continuous poset is called *strongly* Z-*continuous* if the waybelow relation has the interpolation property; that is, $x \ll y$ implies that there exists a $z \in P$ such that $x \ll z \ll y$. If the subset system is union-complete, then any Z-continuous poset is strongly Z-continuous ([V1]). The following table shows the most well known examples of Z-continuous posets. See [V2] for more examples.

Subset system Z	*Z-continuous poset*
All subsets	Completely distributive lattices [R]
Directed subsets	Continuous posets [COMP]
Finite subsets	Unique factoring posets [M]

1. Topology

Definition 1.1. *For a poset P, let $\sigma_Z(P)$ denote the set of all subsets V of P satisfying the following conditions:* (i) $V = \uparrow V$, *and* (ii) *whenever* $\sup S$ *is in* V *for some $S \in Z(P)$, then there exists $s \in S$ such that $s \in V$. Let $\omega_Z(P) = \{P \setminus \uparrow x : x \in P\}$. Let $\lambda(P)$ denote the topology on P generated by $\omega_Z(P) \cup \omega_Z(P)$ as subbasic open sets.*

If Z is the subset system of all subsets, this topology is the same as the interval topology, and if Z is the subset system of all directed subsets, then this topology is the same as the Lawson topology ([COMP]).

Proposition 1.2. *If P is a strongly Z-continuous poset, then $\lambda_Z(P)$ is a T_3 topology.*

PROOF: Since $P \setminus \downarrow x \in \sigma_Z(P)$, $\downarrow x$ is a closed set, and since $P \setminus \uparrow x \in \omega_Z(P)$, $\uparrow x$ is a closed set. Therefore $\{x\} = \uparrow x \cap \downarrow x$ is closed, and hence $\lambda_Z(P)$ is a T_1 topology. Now we shall show that $\lambda_Z(P)$ is regular. It is sufficient if we show that for each $y \in P$, and a subbasic open set U containing y, there exists an open set V such that $y \in V$, and the closure of V is contained in U.

Let $y \in V$ where $V \in \sigma_Z(P)$. Since $y = \sup \Downarrow y$ and $\Downarrow y$ is a Z-ideal, there exists $x \ll y$ such that $x \in V$. Therefore $y \in \Uparrow x \subseteq Cl(\Uparrow x) \subseteq \uparrow x \subseteq V$. Now we shall show that $\Uparrow x$ is an open set. Let $\sup S \in \Uparrow x$ for some Z-set S of P. By the interpolation property, there exists a $z \in P$ such that $x \ll z \ll \sup S$. Then there exists $s \in S$ such that $x \ll z \leq s$. This proves that $\Uparrow x$ is open.

Now let $y \in P \setminus \uparrow x$. Then $x \not\leq y$, and therefore there exists $u \ll x$ such that $u \not\leq y$. By the interpolation property, there exists z such that $u \ll z \ll x$. Therefore $y \in P \setminus \uparrow u \subseteq Cl(P \setminus \uparrow u) \subseteq P \setminus \Uparrow z \subseteq P \setminus \uparrow x$. This completes the proof of the proposition. $\square$

For the remaining of this note, we assume the topology on a Z-continuous P poset is the $\Lambda(P)$ topology. A function between two Z-continuous posets is called a *homomorphism* if it preserves the sups of Z-sets and is an upper adjoint. See [BE] and [V1].

Proposition 1.3. *Let P, Q be Z-continuous posets. If $f : P \to Q$ is a homomorphism, then f is continuous.*

Proof: Since f is an upper adjoint $\inf f^{-1}(\uparrow t)$ exists for all $t \in Q$. Let $s = \inf f^{-1}(\uparrow t)$. Then, since upper adjoints preserves infs, $f(s) = f(\inf f^{-1}(\uparrow t)) = \inf f f^{-1}(\uparrow t) = \inf \uparrow t = t$. Thus $s \in f^{-1}(\uparrow t)$ and hence $f^{-1}(\uparrow t) = \uparrow s$. Therefore $f^{-1}(\uparrow t)$ is closed. Now let $V \in \sigma_Z(Q)$. We shall show that $f^{-1}(V) \in \sigma_Z(P)$. Since f is a monotone map, $f^{-1}(V)$ is an upper set. Let S be a Z-set in P such that $\sup S \in f^{-1}(V)$. Then $f(\sup S) \in V$ and, since f is Z-continuous, $\sup f(S) \in V$. Since $f(S)$ is a Z-set in Q and since $V \in \sigma_Z(Q)$, there exists $x \in S$ such that $f(x) \in V$; that is, $x \in f^{-1}(V)$. Thus $f^{-1}(V) \in \sigma_Z(P)$. This completes the proof that f is continuous. $\square$

The following lemma was proved in [BE].

Lemma 1.4. *Let P, Q be Z-continuous posets, and let (g, d) be a Galois connection from P to Q. If g is Z-continuous, then d preserves the waybelow relation.*

A subposet of a Z-continuous poset is called a subalgebra if the inclusion map is an upper adjoint which preserves the sups of Z-sets. It was shown in [V1] that a subalgebra of a Z-continuous poset is Z-continuous.

Proposition 1.5. *Every subalgebra of a strongly Z-continuous poset P is a closed subspace of P.*

Proof: Let j be the lower adjoint of the inclusion map $i : S \to P$. Let $x \in P \setminus S$. We want to find an open set containing x and contained in $P \setminus S$. Note that $ij(x) \geq x$ which implies that $j(x) > x$. Then there exists $y \in P$ such that $y \not\leq x$ and $y \ll j(x)$. Therefore $x \in P \setminus \uparrow y = V_1$. Since j preserves sups, $y \ll_P j(x) = j(\sup_P \Downarrow x) = \sup_S j(\Downarrow x)$. Then by the above lemma, $j(y) \leq_S j(x) = \sup_S j(\Downarrow x)$ and hence there exists $z \ll x$ such that $j(y) \leq j(z)$. Therefore $x \in \Uparrow z = V_2$. Let $V = V_1 \cap V_2$. We claim $S \cap V = \emptyset$. Indeed, if $r \in S \cap V$, then $y \not\leq r$ and $z \ll r$. Then $y \leq j(y) \leq j(z) \leq j(r) = r$. This contradiction proves the claim. This completes the proof of the proposition. $\square$

A subposet B of a Z-continuous poset P is called a basis if, for all $x \in P$, (i) $\Downarrow x \cap B \in I_Z(P)$ and (ii) $x = \sup \Downarrow x \cap B$ ([V1]).

Proposition 1.6. *If P is a Z-continuous poset with a countable basis, then P is metrizable.*

Proof: Let B be a countable basis of P. We shall show that $\{P \setminus \uparrow b : b \in B\} \cup \{\Uparrow b : b \in B\}$ is a subbasis of the topology. Let $V \in \sigma_Z(P)$ and let $x \in V$. Since $\sup(\Downarrow x \cap B) = x$ and $\Downarrow x \cap B \in I_Z(P)$, there exists $y \in V$ such that $y \in \Downarrow x \cap B$. Then $x \in \Uparrow y \subseteq V$. Now let $P \setminus \uparrow x \in \omega_Z(P)$. Then $P \setminus \uparrow x = P \setminus \uparrow \sup(\Downarrow x \cap B) = P \setminus (\bigcap_{b \in \Downarrow x \cap B} \uparrow b) = \bigcup_{b \in \Downarrow x \cap B} P \setminus \uparrow b$. This proves the claim, and the proposition follows from Urysohn's Metrization Theorem. $\square$

References

[BE] Bandelt H.J., Erné M., *The category of Z-continuous posets*, J. Pure Appl. Algebra **30** (1983), 219–226.

[COMP] Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M., Scott D.S., *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, Heidelberg, and New York, 1980.

[M] Martinez J., *Unique factorization in partially ordered sets*, Proc. Amer. Math. Soc. **33** (1972), 213–220.

[N] Novak D., *Generalization of continuous posets*, Trans. Amer. Math. Soc. **272** (1982), 645–667.

[R] Raney G., *A subdirect-union representation for completely distributive lattices*, Proc. Amer. Math. Soc. **4** (1953), 518–522.

[V1] Venugopalan P., *Z-continuous posets*, Houston J. Math. **12** (1986), 275–294.

[V2] Venugopalan P., *Union complete subset system*, Houston J. Math. **14** (1988), 583–600.

[WWT] Wright J.B., Wagner E.G., Thatcher J.W., *A uniform approach to inductive posets and inductive closure*, Theor. Computer Science **7** (1978), 57–77.

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